

# THE MONIST

## NUMBER: AN INTRODUCTION TO THE THEORY OF ANALYTIC FUNCTIONS

(Continued)

7. We have shown in Chapter II how a given number of numbers, integral or fractional, can be put into one form common to them all (multiples of the same part of the unit); by using this process we were able to compare them, and questions of equality and inequality, of greater than and less than, could be decided. We will now inquire in what manner the same problem can and should be treated for our groups  $((a))$ ; and our discussion will show that all groups  $((a))$  can also be brought into a common standard form.

We have seen in §4 that, if  $\delta$  is a number covered by the group  $((a))$ , an integral number  $n$  always exists such that

$$n \cdot \delta < ((a)) < \text{or} = (n + 1) \cdot \delta.$$

We will first suppose that the group  $((a))$  does not cover the unit 1. Let  $m$  be an integral number greater than 1. In the sequence of numbers  $1/m, 1/m^2, 1/m^3, \dots$ , each number is less than all its predecessors; and, further, by choosing the integral number  $\lambda$  sufficiently great we can make  $1/m^\lambda$  less than any *a priori* given but arbitrarily small number. There exists therefore a first number  $1/m^\lambda$  which is covered by  $((a))$ . Consequently, there exists also a number  $n_\lambda$  (§4), such that

$$\frac{n_\lambda}{m^\lambda} < ((a)) < \text{or} = \frac{n_\lambda + 1}{m^\lambda}.$$

Since now  $1/m^{\lambda+1} > 1/m^{\lambda+2} > 1/m^{\lambda+3} > \dots$  are all covered by  $((a))$ , we obtain an unlimited number of inequalities

$$\frac{n_\mu}{m^\mu} < ((a)) < \text{or} = \frac{n_\mu + 1}{m^\mu}; \mu = \lambda, \lambda + 1, \lambda + 2, \dots$$

It therefore follows that

$$\frac{m \cdot n_\mu}{m^{\mu+1}} < ((a)) < \text{or} = \frac{m \cdot n_\mu + m}{m^{\mu+1}},$$

$$\frac{n_{\mu+1}}{m^{\mu+1}} < ((a)) < \text{or} = \frac{n_{\mu+1} + 1}{m^{\mu+1}};$$

and consequently

$$m \cdot n_\mu < n_{\mu+1} + 1,$$

$$n_{\mu+1} < m \cdot n_\mu + m;$$

$$\text{or} \quad m \cdot n_\mu < \text{or} = n_{\mu+1} < \text{or} = m \cdot n_\mu + m_1,$$

where  $m_1$  is the integral number immediately preceding the number  $m$ .

But it can never be the case that, from a certain index  $\mu$  onwards, we have always

$$m \cdot n_\mu = n_{\mu+1};$$

for, if this were the case, we should have, from this value of  $\mu$  onwards

$$m \cdot n_\mu = n_{\mu+1},$$

$$m^2 \cdot n_\mu = m \cdot n_{\mu+1} = n_{\mu+2},$$

$$\dots \dots \dots$$

$$m^{\mu'} \cdot n_\mu = n_{\mu+\mu'}.$$

Since, on the other hand,  $n_\mu/m^\mu < ((a))$ , we can always choose  $\mu'$  so large that

$$\frac{n_\mu}{m^\mu} + \frac{1}{m^{\mu+\mu'}} < ((a)),$$

or 
$$\frac{m^{\mu'} \cdot n_{\mu} + 1}{m^{\mu + \mu'}} < ((a)),$$

whence it follows that

$$m^{\mu'} \cdot n_{\mu} + 1 < n_{\mu + \mu'} + 1$$

or 
$$m^{\mu'} \cdot n_{\mu} < n_{\mu + \mu'};$$

and this inequality contradicts the hypothesis that, from a certain index, we should have  $m \cdot n_{\mu} = n_{\mu + 1}$ .

We now introduce a sequence of integral numbers  $h_{\mu}$  ( $\mu = \lambda, \lambda + 1, \lambda + 2, \dots$ ) all less than  $m$ , defined by means of the equations

$$h_{\lambda} = n_{\lambda},$$

$$n_{\mu + 1} = m \cdot n_{\mu} + h_{\mu + 1}; \quad \mu = \lambda, \lambda + 1, \lambda + 2, \dots$$

In this sequence, both all numbers  $h_{\mu}$  for which  $\mu$  precedes the number  $\lambda$ , and all numbers  $h_{\mu + 1}$  for which  $n_{\mu + 1} = m \cdot n_{\mu}$ , are lacking. From this definition of the numbers  $h_{\mu}$  it follows that

$$\frac{n_{\lambda + 1}}{m^{\lambda + 1}} = \frac{n_{\lambda}}{m^{\lambda}} + \frac{h_{\lambda + 1}}{m^{\lambda + 1}} = \frac{h_{\lambda}}{m^{\lambda}} + \frac{h_{\lambda + 1}}{m^{\lambda + 1}},$$

$$\frac{n_{\lambda + 2}}{m^{\lambda + 2}} = \frac{n_{\lambda + 1}}{m^{\lambda + 1}} + \frac{h_{\lambda + 2}}{m^{\lambda + 2}} = \frac{h_{\lambda}}{m^{\lambda}} + \frac{h_{\lambda + 1}}{m^{\lambda + 1}} + \frac{h_{\lambda + 2}}{m^{\lambda + 2}},$$

and in general, for  $\mu = \lambda, \lambda + 1, \lambda + 2, \dots$ ,

$$\frac{n_{\mu}}{m^{\mu}} = \frac{h_{\lambda}}{m^{\lambda}} + \frac{h_{\lambda + 1}}{m^{\lambda + 1}} + \frac{h_{\lambda + 2}}{m^{\lambda + 2}} + \dots + \frac{h_{\mu}}{m^{\mu}}.$$

Let us now form from the numbers,  $h_{\nu}/m^{\nu}$ , taken according to the order  $\nu = 1, 2, 3, \dots$ , a group which we will denote by

$$\left( \left( \frac{h_{\nu}}{m^{\nu}} \right) \right);$$

then 
$$\left(\left(\frac{h_v}{m^v}\right)\right) = ((a)).$$

For, if  $p/q$  is a proper fraction which is covered by  $((a))$ , we can clearly choose an integral number  $\mu$  so large that

$$\frac{p}{q} + \frac{1}{m^\mu} < ((a)).$$

On the other hand, we have

$$((a)) < \text{or} = \frac{n_\mu}{m^\mu} + \frac{1}{m^\mu},$$

and therefore

$$\frac{p}{q} < \frac{n_\mu}{m^\mu} = \frac{h_\lambda}{m^\lambda} + \frac{h_{\lambda+1}}{m^{\lambda+1}} + \dots + \frac{h_\mu}{m^\mu} < \left(\left(\frac{h_v}{m^v}\right)\right).$$

That is to say, if  $p/q$  is covered by  $((a))$ , it is covered by  $((h_v/m^v))$ .

If, on the other hand,  $p/q$  is covered by  $((h_v/m^v))$ , then  $\mu$  can always be chosen so large that

$$\frac{p}{q} < \frac{h_\lambda}{m^\lambda} + \frac{h_{\lambda+1}}{m^{\lambda+1}} + \dots + \frac{h_\mu}{m^\mu} = \frac{n_\mu}{m^\mu},$$

and therefore, since  $n_\mu/m^\mu < ((a))$ ,  $p/q$  is covered by  $((a))$ . Thus, every number which is covered by  $((a))$  is covered by  $((h_v/m^v))$  and conversely. Hence

$$((a)) = \left(\left(\frac{h_v}{m^v}\right)\right);$$

as was to be proved.

It was assumed in the above that  $((a))$  does not cover the number 1; if it does so, or if  $N$  is the greatest integral number covered by  $((a))$ , we have only to add the number  $N$  as an element of the group  $((h_v/m^v))$ .

We remark in conclusion that the problem of comparing with one another two differently formed groups, neither of which covers the number 1, has now been simplified by the criteria that two groups  $((h_v/m^v))$  are equal to one another if, and only if, the same  $h_v$  corresponds to the same  $v$  in both the groups, or if, for a certain value of  $v$ , the corresponding  $h_v$  is lacking in one of the groups, then it is lacking in the other also; and that, if there is not equality between the groups, one has only to follow in both the groups the sequence  $h_1, h_2, h_3, \dots$  until one arrives at two different  $h_v$ 's corresponding to the same  $v$  (that group is then the greater to which the greater  $h_v$  belongs), or else until one arrives at an  $h_v$  to which there is nothing in the other group to correspond. If, on the other hand, our groups cover the number 1, that group is the greater which contains the greater integral number as an element.

We have succeeded in transforming the group  $((a))$ , in which no definite order of the elements was assumed, into a new group whose elements are uniquely correlated with the sequence of numbers  $v = 1, 2, 3, \dots$ ; in this correlation, to each  $v$  either one  $h_v$ , or none, corresponds; and further, it cannot be the case that none corresponds to each  $v$  from a certain one onwards. The law of formation of the numbers  $h_v$ , and therefore of the group  $((h_v/m^v))$  also, assumes an unlimited number of operations. This is clearly inevitable if the transformation of  $((a))$  into a group  $((h_v/m^v))$ , defined in a uniform manner, is to embrace the most general groups  $((a))$  possible.

By taking  $m = 10$ , we obtain the familiar representation by means of decimal fractions.

The problem of a uniform representation of groups  $((a))$  may be treated, as is easily seen, in many different ways. One of the best-known is representation by continued fractions. In the preceding work, as also in the two

following paragraphs, in which it will be our task to deduce the profound theorems of Weierstrass and Cantor (enunciated already in § 6), we have however preferred to follow the method of proof of Weierstrass himself, and to use the form which he himself chose for the representation of groups  $((a))$ .

The task of obtaining for diverse mathematical combinations (or what comes to the same thing, for diverse combinations of numbers) a uniform mode of representation of the widest possible scope comes very near to coinciding with the general task of mathematics itself; and with it mathematics as a science stands or falls. We find, moreover, in every true science a like task—that of embracing under uniform aspects as many data of knowledge as possible.

8. We return now to the following theorem, enunciated in § 6: "Let  $A$  and  $B$  ( $A < B$ ) be two numbers, which may be arbitrarily near to one another. Between  $A$  and  $B$  there always exists a group  $((a))$ ,  $A < ((a)) < B$ , which is not equal either to an integral or to a fractional number."

The method of proof coincides almost completely with that used by Weierstrass in his deduction of the theorem on *limiting-values* (cf. § 6), but our exposition gains in lucidity if we first prove the theorem just enunciated.

We denote by  $a_v$  ( $v = 1, 2, 3, \dots$ ) the aggregate of all integral and fractional numbers, and suppose them so correlated with the sequence of all integral numbers  $1, 2, 3, \dots$ , that to every integral number  $v$  corresponds one, and only one, integral or fractional number  $a_v$ ; and, conversely, to every number  $a_v$  corresponds one, and only one, number  $v$  (Cf. Chap. II, § 7).

In every interval  $[A < B]$ , however small it may be, there always exist an unlimited sequence of numbers  $a$  defined as above (Chap. II, § 7). We begin now with a

definite interval  $[A < B]$  and denote by  $a_{\theta_1}$  that one of the numbers  $a_v$  which lie in the interval  $[A < B]$  which has the lowest index  $v = \theta_1$ ; i. e.,  $a_{\theta_1}$  is the first of the numbers  $a_v$  which belongs to the interval. We therefore have

$$A < \text{or} = a_{\theta_1} < \text{or} = B \text{ and } \theta_1 > \text{or} = 1,$$

where, if  $\theta_1 > 1$ , none of the numbers  $a_1, a_2, \dots, a_{\theta_1-1}$  belongs to the interval  $[A < B]$ . We now divide the interval  $[A < B]$  into  $m (> \text{or} = 3)$  equally large intervals

$$[A < A + \frac{1}{m} (B - A)],$$

$$[A + \frac{1}{m} (B - A) < A + \frac{2}{m} (B - A)]$$

$$\dots \dots \dots$$

$$[A + \frac{m-1}{m} (B - A) < B].$$

Among these intervals there must necessarily be a first one

$$[A + \frac{n_1}{m_1} (B - A) < A + \frac{n_1+1}{m} (B - A)],$$

where  $0 < \text{or} = n_1 < \text{or} = m - 1$ ,<sup>10</sup> which does not include  $a_{\theta_1}$ , and therefore includes none of the numbers  $a_1, a_2, \dots, a_{\theta_1}$ .

Now let  $a_{\theta_2}$  be the first of the numbers  $a_v$  which belongs to this interval, and for which therefore we have

$$A + \frac{n_1}{m} (B - A) < \text{or} = a_{\theta_2} < \text{or} = A + \frac{n_1+1}{m} (B - A),$$

where  $0 < \text{or} = n_1 < \text{or} = m - 1$  and  $\theta_2 > \theta_1$ ;

<sup>10</sup> Although the symbol  $0$  has not yet been defined, we wish, for brevity to use the notation  $0 < \text{or} = n_1$ ;  $0 < n_1$  will mean that  $n_1$  is an integral number;  $0 = n_1$  will mean that  $n_1$  does not occur at all, and that, in this case,

$$A + \frac{n_1}{m} (B - A) = A \text{ and } A + \frac{n_1+1}{m} (B - A) = A + \frac{1}{m} (B - A),$$

and that  $n_1$ , etc., do not occur.

and we divide also the interval so obtained into  $m$  equally large parts. Among the new sub-intervals so arising there again necessarily exists a first one,

$$\left[A + \frac{n_2}{m^2} (B - A) < A + \frac{n_2 + 1}{m^2} (B - A)\right],$$

where  $m \cdot n_1 < \text{or} = n_2 < \text{or} = m \cdot n_1 + m - 1$ , which does not include  $a_{\theta_2}$ , and therefore includes none of the numbers  $a_1, a_2, \dots, a_{\theta_1}, a_{\theta_1+1}, \dots, a_{\theta_2}$ .

By carrying on this process we obtain an interval,

$$\left[A + \frac{n_\mu}{m^\mu} (B - A) < A + \frac{n_\mu + 1}{m^\mu} (B - A)\right],$$

where  $m \cdot n_{\mu-1} < \text{or} = n_\mu < \text{or} = m \cdot n_{\mu-1} + m - 1$ , which includes none of the numbers

$$a_1, a_2, \dots, a_{\theta_1}, \dots, a_{\theta_2}, \dots, a_{\theta_\mu} \\ (1 < \text{or} = \theta_1 < \theta_2 < \dots < \theta_\mu),$$

and where  $a_{\theta_{\mu+1}}$  is the first of the numbers  $a$ , which occurs in the interval. Each new interval is a part of its immediate predecessor. It can, however, never be the case that from a definite  $\mu$  onwards, from, for example,  $\mu = \lambda$ , one of the two equations  $n_{\mu+1} = m \cdot n_\mu$  or  $n_{\mu+1} = m \cdot n_\mu + m - 1$  always holds. For if this were the case, then either

$$A + \frac{n_\mu}{m^\mu} (B - A) \text{ or } A + \frac{n_\mu + 1}{m^\mu} (B - A)$$

would, from  $\mu = \lambda$  onwards, always represent the same number,  $C$  say. But the number  $C$  is again one of our numbers  $a$ , e.g.,  $a_v$ . Since  $\mu < \text{or} = \theta_\mu$  always, and since none of the numbers  $a_1, a_2, \dots, a_{\theta_\mu}$  belong to the interval

$$\left[A + \frac{n_\mu}{m^\mu} (B - A) < A + \frac{n_\mu + 1}{m^\mu} (B - A)\right],$$

it follows that  $C$  cannot belong to any such interval so soon

as  $\mu > \bar{\nu}$ . Therefore we have

$$\frac{n_\mu}{m^\mu} > \frac{n_\lambda}{m^\lambda}$$

providing only that  $\mu$  is taken sufficiently large. Thus, neither of the equations  $n_{\mu+1} = m \cdot n_\mu$  or  $n_{\mu+1} = m \cdot n_\mu + m - 1$  can always hold from a certain number  $\mu$  onwards.

We now return to the method of proof of the preceding section. We put

$$\begin{aligned} n_1 &= h_1 < \text{or} = m - 1, \\ n_{\mu+1} &= m \cdot n_\mu + h_{\mu+1}; & h_{\mu+1} < \text{or} = m - 1; \\ \mu &= 1, 2, 3, \dots, \end{aligned}$$

where the number  $h_{\mu+1}$  is lacking only in those cases where  $n_{\mu+1} = m \cdot n_\mu$ . Neither this relation nor the relation

$$h_{\mu+1} = m - 1$$

can, as we have seen, hold for all  $\mu$ 's from a certain one onwards.

We now form the group  $((h_\nu/m^\nu)) = 1, 2, 3, \dots$ . The group

$$A + \left( \left( \frac{h_\nu}{m^\nu} \right) \right) (B - A) = \left( \left( A, \frac{h_\nu}{m^\nu} (B - A) \right) \right)^{11}$$

covers all numbers

$$A + \left( \frac{h_1}{m} + \frac{h_2}{m^2} + \dots + \frac{h_\mu}{m^\mu} \right) (B - A) = A + \frac{n_\mu}{m^\mu} (B - A)$$

and we have

$$A + \left( \left( \frac{h_\nu}{m^\nu} \right) \right) (B - A) > A + \frac{n_\mu}{m^\mu} (B - A),$$

however great the index  $\mu$  may be chosen. Since the equation

<sup>11</sup> By this we mean a group  $((a))$  whose elements are the number  $A$  and the numbers  $\frac{h_\nu}{m^\nu} (B - A)$ ;  $\nu = 1, 2, 3, \dots$

$$\frac{h_{\mu+1}}{m^{\mu+1}} + \frac{h_{\mu+2}}{m^{\mu+2}} + \dots + \frac{h_{\mu+\mu'}}{m^{\mu+\mu'}} = \frac{m-1}{m^{\mu+1}} \left(1 + \frac{1}{m} + \dots + \frac{1}{m^{\mu'-1}}\right)$$

cannot hold for all values of  $\mu'$  greater than a fixed number, and since therefore, if only  $\mu'$  be chosen sufficiently great, there always exists a number  $\bar{\mu} < \mu'$  which is so great that  $h_{\mu+\bar{\mu}} < m-1$ , we therefore have for all such values of  $\mu'$

$$\begin{aligned} & \frac{h_{\mu+1}}{m^{\mu+1}} + \frac{h_{\mu+2}}{m^{\mu+2}} + \dots + \frac{h_{\mu+\mu'}}{m^{\mu+\mu'}} \\ & < \text{or} = \frac{m-1}{m^{\mu+1}} \left(1 + \frac{1}{m} + \dots + \frac{1}{m^{\mu'-1}}\right) - \frac{1}{m^{\mu+\bar{\mu}}} \\ & < \text{or} = \frac{1}{m^{\mu}} \left(1 - \frac{1}{m^{\mu'}}\right) - \frac{1}{m^{\mu+\bar{\mu}}} \\ & < \frac{1}{m^{\mu}} - \frac{1}{m^{\mu+\bar{\mu}}} ; \end{aligned}$$

and therefore—and that however large  $\mu'$  may be chosen—

$$\frac{n_{\mu}}{m^{\mu}} + \frac{h_{\mu+1}}{m^{\mu+1}} + \dots + \frac{h_{\mu+\mu'}}{m^{\mu+\mu'}} < \frac{n_{\mu}+1}{m^{\mu}} - \frac{1}{m^{\mu+\bar{\mu}}}.$$

Hence it follows again that

$$\left(\left(\frac{h_v}{m^v}\right)\right) < \frac{n_{\mu}+1}{m^{\mu}},$$

and consequently

$$\begin{aligned} A + \frac{n_{\mu}}{m^{\mu}} (B-A) & < A + \left(\left(\frac{h_v}{m^v}\right)\right) (B-A) \\ & < A + \frac{n_{\mu}+1}{m^{\mu}} (B-A); \end{aligned}$$

and this relation still holds good however great the number  $\mu$  may be chosen.

From this it follows again that the number

$$A + \left(\left(\frac{h_v}{m^v}\right)\right) (B-A),$$

cannot be equal either to a fractional or to an integral number  $a_v$ ; for if we choose  $\mu > \text{or} = \bar{\nu}$ , all the numbers  $a_v$  whose indices are less than or equal to  $\theta_\mu$ , which, in turn, is greater than or equal to  $\mu$ , lie outside the interval

$$\left[ A + \frac{n_\mu}{m^\mu} (B - A) < A + \frac{n_\mu + 1}{m^\mu} (B - A) \right],$$

as was to be proved.

We have thus established the existence of a group

$$A + \left( \left( \frac{h_\nu}{m^\nu} \right) \right) (B - A),$$

which is not equal either to an integral or to a fractional number and which lies between two arbitrarily close given numbers. The course of our proof has been such that we obtained an arithmetical representation of the number

$$A + \left( \left( \frac{h_\nu}{m^\nu} \right) \right) (B - A).$$

It is a characteristic feature of the Weierstrassian theory of functions and one that we shall meet again on every side, that Weierstrass always endeavors so to carry through the proof of the existence of any particular mathematical entity that he deduces for it a definite mode of arithmetical representation.

9. We now proceed to Weierstrass' theorem (§6).

We have seen that by a *limiting-value* is to be understood a group such that if it be enclosed between two limits (i. e., two other groups) which are brought arbitrarily near to one another, there always exists between these two limits an unlimited number of other groups.

Let now  $A$  and  $B$ , where  $A < B$ , be two groups  $((a))$ , of which we assume that between them there lie an unlimited number of other groups  $((a))$ . Just as in the preceding paragraph, we now divide the interval  $[A < B]$  into

$m$  ( $m >$  or  $= 2$ ) equal parts

$$[A < A + \frac{1}{m} (B - A)];$$

$$[A + \frac{1}{m} (B - A) < A + \frac{2}{m} (B - A)];$$

. . . . .

$$[A + \frac{m-1}{m} (B - A) < B].$$

There exists then a first interval

$$[A < A + \frac{1}{m} (B - A)]$$

or  $[A + \frac{n_1}{m} (B - A) < A + \frac{n_1+1}{m} (B - A)],$

where  $1 <$  or  $= n_1 <$  or  $= m - 1$ , which contains an unlimited number of groups  $((a))$ . If we divide this interval again into  $m$  parts, there exists similarly among the new intervals so arising a first one

$$[A + \frac{n_2}{m^2} (B - A) < A + \frac{n_2+1}{m^2} (B - A)],$$

where  $m.n_1 <$  or  $= n_2 <$  or  $= m.n_1 + m - 1$ , which contains an unlimited number of groups  $((a))$ . Continuing in this manner we obtain a sequence of intervals

$$[A + \frac{n_\mu}{m^\mu} (B - A) < A + \frac{n_\mu+1}{m^\mu} (B - A)],$$

where  $m.n_\mu <$  or  $= n_{\mu+1} <$  or  $= m.n_\mu + m - 1$ ,  
and  $\mu = 1, 2, 3, \dots$ ,

such that each successive one is a part of its predecessor, and which all contain an unlimited number of groups  $((a))$ .

The number

$$(A + ((\frac{h_\nu}{m^\nu})) (B - A))$$

where  $h_{\mu+1} + m.n_{\mu} = n_{\mu+1}$ , is included in all these intervals. Since we can make the interval arbitrarily small by sufficiently increasing the number  $\mu$ , it follows that the number

$$A + \left( \left( \frac{h_{\nu}}{m^{\nu}} \right) \right) (B - A),$$

is a limiting-value, so that our theorem is proved. In contrast with the applications of Weierstrass' method of deduction which we made in §§ 7 and 8 there is, however, the essential distinction that here it may happen that in the number

$\left( \left( \frac{h_{\nu}}{m^{\nu}} \right) \right)$ , all the fractions from a certain  $\nu$  onwards are

lacking. And in this way the group obtained as a limiting-value may be either equal to or unequal to an integral or fractional number.

10. We now proceed to demonstrate how the rules of calculation for numbers can be made equally valid for our groups  $((a))$ . We have seen that these groups are strictly defined number-objects which, including, as they do, also the integral and fractional numbers, possess a reality which is given by and with the reality of numbers themselves. If we can now establish further that the rules of calculation with numbers apply unaltered to them,—so that we can calculate with groups  $((a))$  in precisely the same way as with numbers—we shall thus have shown that they are numbers, and indeed numbers in a sense essentially wider than that as yet known to us.

We begin with the concept of the *sum* of groups. This concept yields itself almost immediately. For it is clear that a group which is formed from elements  $a + b$ , where  $a$  typifies the elements of the group  $((a))$  and  $b$  those of the group  $((b))$ , is itself a group  $((a + b))$ . For a number  $l$  always exists, so large as to be covered by neither  $((a))$  nor  $((b))$ , whence it follows that the number  $2l$

cannot be covered by a group formed from the elements  $a + b$ , so that these form a group  $((a + b))$ . Consequently we have as the definition of sum:

"By the sum of two groups  $((a))$  and  $((b))$  we mean the group  $((a + b))$ ."

We obtain at once the following theorems:

" $((a)) + [((b)) + ((c))] = [((a)) + ((b))] + ((c))$ ," the associative law;

" $((a)) + ((b)) = ((b)) + ((a))$ ," the commutative law; and

"In the summation of several groups  $((a))$  the order of summing is indifferent."

"If  $((a)) + ((b)) = ((c))$ , then  $((a)) < ((c))$  and  $((b)) < ((c))$ ."

"If  $((a)) = ((b))$  and  $((c)) = ((d))$ , then  $((a)) + ((c)) = ((b)) + ((d))$ ."

"If  $((a)) > ((b))$  and  $((c)) > ((d))$ , then  $((a)) + ((c)) > ((b)) + ((d))$ ."

"If  $((a)) = ((b))$  and  $((c)) > ((d))$ , then  $((a)) + ((c)) > ((b)) + ((d))$ ."

"If  $((a)) = ((b))$  and  $((a)) + ((c)) > ((b)) + ((d))$ , then  $((c)) > ((d))$ ."

"If  $((a)) < ((b))$  and  $((a)) + ((c)) = ((b)) + ((d))$ , then  $((c)) > ((d))$ ."

"If  $((a)) = ((b))$  and  $((a)) + ((c)) = ((b)) + ((d))$ , then  $((c)) = ((d))$ ."

The same laws which hold for the addition of numbers thus hold in unaltered form for the addition of groups  $((a))$ .

Hence it follows that groups  $((A))$ , whose elements are themselves groups  $((a))$ , may be treated in the same way as groups whose elements are numbers.

For let  $((A))$  be a group whose elements are groups  $((a))$ . In analogy with § 1 we then have:

"A number or a group  $l$  is said to be *covered* by a group which is formed from the elements  $((a))$  if a number of elements can be selected from the group whose sum is greater than  $l$ .

"The elements  $A = ((a))$  form a group  $((A))$  if a number  $l$  exists which is not covered by the group."

"Between two groups  $((((a))))$  and  $((((b))))$ , whose elements  $((a))$  and  $((b))$  are themselves again groups, there is equality,

$$(((a))) = (((b))), (((b))) = (((a)))$$

if every number which is *covered* by the one group is *covered* by the other also; on the other hand, we have

$$(((a))) > (((b))), (((b))) < (((a)))$$

if numbers exist which are covered by  $((((a))))$  without being covered by  $((((b))))$ ."

If we now introduce, in place of numbers  $a$ , groups  $((a))$ , the developments given in § 7, whereby our groups  $((a))$  were reduced to the standard form  $((\frac{h_v}{m^v}))$  may be retained word for word; i. e., "*The universe of numbers contains no other groups than such as can be represented in the form  $((\frac{h_v}{m^v}))$ .*"

11. Having introduced the concept of the sum of two groups we easily see how the theorem of Georg Cantor (§ 6) is included in the chain of proof developed in § 8. We there started from the sequence of all integral and fractional numbers  $a_v$ , which we had uniquely correlated with the sequence of all integral numbers, 1, 2, 3, . . . . . If we introduce, in place of the numbers  $a_v$ , groups  $((a))$ , which are distinct from each other and which form an unlimited

sequence,  $((a))_1, ((a))_2, ((a))_3, \dots$ , uniquely correlated with the integral numbers 1, 2, 3,  $\dots$ , all our deductions remain the same as in the case in which we had to deal with integral and fractional numbers  $a_v$ .

*We can always form a group  $((a))$  which is equal to none of the groups  $((a))_v$ , and which lies between any two of them.*

Consequently the groups  $((a))$  cannot, as could the integral and fractional numbers, be written down in an ordered sequence, uniquely correlated with the integral numbers 1, 2, 3,  $\dots$ . On the other hand, the totality of all the groups  $((a)) < \text{or} = 1$  is obtained by letting the numbers  $h_v$  in the expression  $((\frac{h_v}{m^v}))$  (cf. §7) each assume in turn all integral values  $< \text{or} = m - 1$  or else vanish from the expression (without letting them vanish either altogether or from a certain  $v$  onwards)

By the introduction of groups  $((a))$  we have hit upon an infinity of a higher kind than the intuitively given infinity which meets us in the totality of all integral numbers 1, 2, 3,  $\dots$ . We have arrived at the concept of the continuum.

*"The continuum is the totality of all groups  $((a))$ ."*

"The groups forming the continuum cannot all be uniquely correlated with the number-sequence 1, 2, 3,  $\dots$ "

12. We have next to examine whether the theorem about the *difference* of two numbers (Chap. I, §6) can also be extended to the *difference* of two groups. As will be easily shown, this is in fact the case.

"If  $((b)) > ((a))$ , a new group  $((c))$  can always be formed such that  $((b)) = ((a)) + ((c))$ ."

Let  $((b)) = h' + ((\frac{h'_v}{m^v}))$  and  $((a)) = h + ((\frac{h_v}{m^v}))$  (cf. §7), where  $h$  and  $h'$  are integral numbers of which

either  $h$  alone or  $h'$  and  $h$  simultaneously may be lacking, where  $h' < \text{or} = m - 1$  and  $h < \text{or} = m - 1$ , and where  $h$ , and  $h'$ , may, for certain indices, be lacking in their respective groups, provided however that neither of them is always lacking from a certain least index onwards.

Since  $((b)) > ((a))$ , a number  $l$  always exists which is covered by  $((b))$  without being covered by  $((a))$ . Thus  $l > h + ((\frac{h_v}{m^v}))$ , and therefore

$$l > h + \frac{h_1}{m} + \frac{h_2}{m^2} + \dots + \frac{h_v}{m^v} = \frac{n_v}{m^v} > \text{or} = \frac{1}{m^v};$$

and, if  $\lambda$  be taken sufficiently large,

$$h' + \frac{h'_1}{m} + \frac{h'_2}{m^2} + \dots + \frac{h'_v}{m^v} = \frac{n'_v}{m^v} > l,$$

where  $v = \lambda, \lambda + 1, \lambda + 2, \dots$ ; whence it follows that

$$\frac{n'_v}{m^v} > \frac{n_v}{m^v}; v = \lambda, \lambda + 1, \lambda + 2, \dots \text{ or}$$

$$n'_v - 1 > \text{or} = n_v > \text{or} = 1.$$

On the other hand,  $1 = ((\frac{m-1}{m^v}))$ , and the group  $((b))$

can thus be represented by

$$\begin{aligned} ((b)) &= \frac{n'_\lambda - 1}{m^\lambda} + \frac{1}{m^\lambda} + ((\frac{h'_{\lambda+\mu}}{m^{\lambda+\mu}})) \\ &= \frac{n'_\lambda - 1}{m^\lambda} + ((\frac{m-1}{m^{\lambda+\mu}})) + ((\frac{h'_{\lambda+\mu}}{m^{\lambda+\mu}})) \\ &= \frac{n'_\lambda - 1}{m^\lambda} + ((\frac{h'_\lambda + m - 1}{m^{\lambda+\mu}})); \mu = 1, 2, 3, \dots \end{aligned}$$

Further,

$$h'_v + m - 1 > \text{or} = h_v \text{ and } h'_v + m - 1 < \text{or} = 2(m - 1).$$

If we now introduce a series of equations

$$\begin{aligned} n_\lambda + k_\lambda &= n'_\lambda - 1, \\ h_{\lambda+\mu} + k_{\lambda+\mu} &= h'_{\lambda+\mu} + m - 1, \end{aligned}$$

we obtain in  $\frac{k_\lambda}{m^\lambda} + ((\frac{k_{\lambda+\mu}}{m^{\lambda+\mu}}))$  the required group  $((c))$ .

$$\begin{aligned}\text{For } ((b)) &= \frac{n'_\lambda - 1}{m^\lambda} + ((\frac{h'_{\lambda+\mu} + m - 1}{m^{\lambda+\mu}})) \\ &= \frac{n_\lambda}{m^\lambda} + ((\frac{h_{\lambda+\mu}}{m^{\lambda+\mu}})) + \frac{k_\lambda}{m^\lambda} + ((\frac{k_{\lambda+\mu}}{m^{\lambda+\mu}})) = ((a)) + ((c)).\end{aligned}$$

Certain of the numbers  $k_\lambda, k_{\lambda+\mu}, \dots$  may be lacking in the expression for  $((c))$ , but not all of them from a certain index onwards; for, if this were so,  $((\frac{h'_{\lambda+\mu}}{m^{\lambda+\mu}}))$  would not contain an unlimited number of elements.

The group  $((\frac{k_{\lambda+\mu}}{m^{\lambda+\mu}}))$  is not a group of the type which we have denoted by  $((\frac{h_\nu}{m^\nu}))$ , but in accordance with § 7, it can always be transformed either into such a group or into the sum of such a group and an integral number.

That the group  $((c))$  is uniquely determined follows from § 10.

13. We wish now to examine what is to be understood by the product of groups  $((a))$ . Let  $((a))$  be a group and  $m$  a number. Since  $((a))$  is a group, a number  $l$  always exists such that

$$\sum_{v=1}^{v=n} a_v < l,$$

however great  $n$  may be chosen. Hence

$$\sum_{v=1}^{v=n} m \cdot a_v < m \cdot l,$$

however great  $n$  may be chosen. The group formed from the elements  $m \cdot a$  is thus a group  $((m \cdot a)) = ((a \cdot m))$ .

"By the product of the number  $m$  and the group  $((a))$  we mean the group  $((m.a)) = ((m.a))$ ; and we write

$$m.((a)) = ((a)).m = ((m.a)) = ((a.m))."$$

We obtain at once

$$\left[ \sum_{v=1}^{v=n} m_v \right].((a)) = \sum_{v=1}^{v=n} m_v.((a)) = \sum_{v=1}^{v=n} ((m_v.a))."$$

Now let  $((a))$  and  $((b))$  be two distinct groups. There always exists a number  $l$  such that

$$\sum_{v=1}^{v=n} a_v < l,$$

however great the number  $n$  may be chosen. Further, there always exists a number  $m$  such that  $((b)) < m$ . We therefore have

$$\sum_{v=1}^{v=n} a_v.((b)) = \left[ \sum_{v=1}^{v=n} a_v \right].((b)) < m.l.$$

Consequently a group formed from the elements  $a.((b)) = ((b)).a$  is a group  $((a.((b)))) = (((b)).a)$ , and similarly one formed from the elements  $b.((a)) = ((a)).b$  is a group  $((b.((a)))) = (((a)).b)$ .

Let now  $l$  be a number covered by the group  $((a.((b))))$ . There therefore always exists a number  $n_1$  so great that

$$\sum_{v=1}^{v=n_1} a_v.((b)) > l;$$

and hence also

$$\sum_{v=1}^{v=n_1} a_v.((b)) > l + \delta,$$

if  $\delta$  is taken sufficiently small. On the other hand, there

always exists a number  $n_2$  so great that

$$\sum_{v=1}^{v=n_1} a_v \cdot ((b))^{(n_2+1)} < \delta. \quad 12$$

Hence

$$\sum_{v=1}^{v=n_1} a_v \cdot \left[ \sum_{\mu=1}^{\mu=n_2} b_\mu + ((b))^{(n_2+1)} \right] > l + \delta,$$

whence follows

$$\sum_{v=1}^{v=n_1} a_v \left[ \sum_{\mu=1}^{\mu=n_2} b_\mu \right] > l \text{ or } \sum_{\mu=1}^{\mu=n_2} b_\mu \cdot \sum_{v=1}^{v=n_1} a_v > l$$

Hence it follows again that

$$\sum_{\mu=1}^{\mu=n_2} b_\mu \cdot ((a)) > l.$$

"The number  $l$  which is covered by  $((a \cdot ((b))))$  is always covered by  $((b \cdot ((a))))$  also, and accordingly we have the equation

$$((a \cdot ((b)))) = ((b \cdot ((a))))."$$

"By the product of the group  $((a))$  and the group  $((b))$  or  $((a)) \cdot ((b))$ , we mean the group

$$((a \cdot ((b)))) = ((b \cdot ((a))))."$$

We have already seen that the Commutative Law,

$$((a)) \cdot ((b)) = ((b)) \cdot ((a)),$$

holds.

A simple consideration shows that the same is true of the Associative Law, or that

$$[[((a)) \cdot ((b))] \cdot ((c))] = ((a)) \cdot [((b)) \cdot ((c))]."$$

<sup>12</sup> For the notation  $((b))^{(n_2+1)}$ , cf. § 4.

It is seen likewise that, in the multiplication of a finite number of groups  $((a))$ , the order in which the multiplications are performed is immaterial, so that the same number is always obtained as the final result.

We have further:

"If  $((a)) = ((b))$ , and  $((c)) = ((d))$ , then

$$((a)).((c)) = ((b)).((d))."$$

"If  $((a)) > ((b))$ , and  $((c)) > ((d))$ , then

$$((a)).((c)) > ((b)).((d))."$$

Also, conversely,

"If  $((a)) = ((b))$ , and  $((a)).((c)) > ((b)).((d))$ , then  $((c)) > ((d))$ ."

"If  $((a)) = ((b))$ , and  $((a)).((c)) = ((b)).((d))$ , then  $((c)) = ((d))$ ."

"If  $((a)) > ((b))$ , and  $((a)).((c)) = ((b)).((d))$ , then  $((c)) < ((d))$ ."

14. We have shown how the laws of *addition* (§10), *subtraction* (§12), and *multiplication* (§13) hold unchanged for our groups  $((a))$ . The concept of division also holds unchanged.

"If  $((b))$  and  $((a))$  are two groups we can always form a new group  $((c))$ , such that

$$((b)) = ((a)).((c))."$$

For let  $m$  be an integral number which is not covered by  $((a))$ . A group  $((\alpha))$  always exists such that (§12)  $m = ((a)) + ((\alpha))$ ; and, therefore, such that

$$1 = ((a))/m + ((\alpha))/m.$$

We now introduce the new group

$$((\beta)) = 1 + \left( \left\{ ((a))/m \right\}^v \right); \quad v = 1, 2, 3, \dots;$$

and we have

$$\frac{((a))}{m} \cdot ((\beta)) = \left( \left( \left\{ \frac{((a))}{m} \right\}^v \right) \right); \quad v = 1, 2, 3, \dots;$$

$$\text{so that } ((\beta)) = 1 + \frac{((a))}{m} \cdot ((\beta)).$$

By adding  $\frac{((a))}{m} \cdot ((\beta))$  to each side, we obtain

$$\begin{aligned} ((\beta)) + \frac{((a))}{m} \cdot ((\beta)) &= 1 + \left\{ \frac{((a))}{m} + \frac{((a))}{m} \right\} \cdot ((\beta)) \\ &= 1 + ((\beta)); \end{aligned}$$

and hence it follows that

$$\frac{((a))}{m} \cdot ((\beta)) = 1.$$

$$\text{Thus } ((a)) \cdot \frac{((b))}{m} \cdot ((\beta)) = ((b)), \text{ or}$$

$$((c)) = \frac{((b))}{m} \cdot ((\beta)).$$

We have now shown that all the laws for *addition*, *subtraction*, *multiplication* and *division*, which we deduced in Chapters I and II for integral and fractional numbers, hold also for our groups  $((a))$  in unchanged form.

The groups  $((a))$ , which include the integral and fractional numbers and have, at the same time, a much wider scope than they, are themselves numbers. They may rightly be called *general numbers*. The customary name for them, when they are neither integral nor fractional numbers, is *incommensurable numbers*. Since for long, and indeed until quite a recent time, they were regarded not as conceptionally given objects, but either as geometrical magnitudes arrived at by intuitive perception or as symbols which are submitted to the rules of operation valid for true numbers, the name of *irrational numbers* has gained currency and

remains still in common use. In the terminology still in use the integral and fractional numbers are then classed together under the antithetical name of *rational numbers*.<sup>13</sup>

After we have now shown that  $((a))$  may be dealt with in just the same way as those numbers  $a$  which we have hitherto called *absolute numbers*, we will henceforward include under this name the numbers  $((a))$  also. The general number of a number-system with unit  $e$  will thus be denoted by  $e \cdot ((a))$ ; it obeys the same rules of calculation as does  $e \cdot a$ , where  $a$  is a rational number.

15. The historically oldest problem which, after thousands of years of more or less conscious seeking, has been given a real solution by the introduction of the groups  $((a))$ , or *general numbers*, may well be that of the conceptual meaning of the fact of geometrical experience that the side and diagonal of a square cannot be measured by any common measure. In trying to find a number  $x$  which satisfies the equation  $x^2 = 2$ , one finds at once that no such number (integral or fractional number) can exist; but geometrical intuition nevertheless gives us for the required unknown, for  $x$ , a certain something. How then is this something to be conceived? What precedes has shown us how to answer this question.

We form by arithmetical processes the group  $((a))$  or, in other words, the general number which represents our required  $x$ . This can most simply be effected by following out the same train of reasoning as in §7.

Let then  $P$  be an integral number which is not equal to the product obtained by multiplying by itself any integral number greater than 1. Also let  $m$  be an integral number greater than 1. To the number  $m^{2v}$  ( $v = 1, 2, 3, \dots$ ) there always corresponds then an integral number  $n_v > 1$ , such that

<sup>13</sup> This is not the derivation usually given; irrational numbers are so-called because they cannot be represented by a ratio, i. e., by a fraction.—Editor.

$$\left[ \frac{n_v}{m^v} \right]^2 < P < \left[ \frac{n_v + 1}{m^v} \right]^2;$$

and hence also an integral number  $n_{v+1}$ , such that

$$\left[ \frac{n_{v+1}}{m^{v+1}} \right]^2 < P < \left[ \frac{n_{v+1} + 1}{m^{v+1}} \right]^2.$$

We see that  $[n_{v+1} + 1]^2 > [n_v \cdot m]^2$ ; and therefore

$$n_{v+1} + 1 > n_v \cdot m \text{ or } n_{v+1} > \text{or} = n_v \cdot m.$$

On the other hand, we have also  $n_v \cdot m + m > n_{v+1}$ , and therefore  $n_v \cdot m + m - 1 > \text{or} = n_{v+1}$ . We now introduce integral numbers  $h, h_1, h_2, h_3, \dots$ , defined as follows:

$$h^2 < P < (h + 1)^2,$$

$$h_1 + n \cdot m = n_1,$$

$$h_2 + n_1 \cdot m = n_2,$$

$$\dots \dots \dots$$

$$h_v + n_{v-1} \cdot m = n_v \quad (v = 1, 2, 3, \dots)$$

The numbers  $h_1, h_2, h_3, \dots$  are such that whenever  $n_{v-1} \cdot m \neq n_v$ , then  $h_v < \text{or} = m - 1$ ; and, further, when  $n_v \cdot m = n_{v+1}$ , there is no  $h_{v+1}$  corresponding to  $v$ .

In addition, it can never occur that from a certain  $v$  onwards, say from  $v = \lambda$ ,  $n_v \cdot m = n_{v+1}$ .

For since

$$\left[ \frac{n_\lambda}{m^\lambda} \right]^2 < P < \left[ \frac{n_{\lambda+\mu}}{m^{\lambda+\mu}} + \frac{1}{m^{\lambda+\mu}} \right]^2;$$

and since  $\mu$  can always be chosen so great that

$$\left[ \frac{n_\lambda}{m^\lambda} + \frac{1}{m^{\lambda+\mu}} \right]^2 < P \text{ or } \left[ \frac{m^\mu n_\lambda + 1}{m^{\lambda+\mu}} \right]^2 < P,$$

it follows that  $m^\mu \cdot n_\lambda + 1 < n_{\lambda+\mu} + 1$ , or  $m^\mu \cdot n_\lambda < n_{\lambda+\mu}$ ; and

this contradicts the hypothesis that  $m^\mu \cdot n_\nu = n_{\lambda+\mu}$ , or that the equation  $m \cdot n_\nu = n_{\nu+1}$  should be satisfied from  $\nu = \lambda$  onwards.

We now form the group  $S = h + (\frac{h_\nu}{m^\nu})$ .

Since all the  $h_\nu$ 's from a certain  $\nu$  onward cannot be lacking, we have

$$h + \frac{h_1}{m} + \frac{h_2}{m^2} + \dots + \frac{h_\nu}{m^\nu} < S.$$

On the other hand

$$h + \frac{h_1}{m} + \frac{h_2}{m^2} + \dots + \frac{h_\nu}{m^\nu} = \frac{n_\nu}{m^\nu}.$$

Further,

$$\frac{h_{\nu+1}}{m^{\nu+1}} + \frac{h_{\nu+2}}{m^{\nu+2}} + \dots + \frac{h_{\nu+\mu}}{m^{\nu+\mu}} < (m-1) \cdot \frac{1 - 1/m^\mu}{1 - 1/m} \cdot \frac{1}{m^{\nu+1}} < 1/m^\nu;$$

and therefore

$$\frac{n_\nu}{m^\nu} < S < \text{or} = \frac{n_\nu + 1}{m^\nu}, \text{ i. e., } \left[ \frac{n_\nu}{m^\nu} \right]^2 < S^2 < \text{or} = \left[ \frac{n_\nu + 1}{m^\nu} \right]^2$$

We have therefore

$$P = \left[ \frac{n_\nu}{m^\nu} \right]^2 + \delta \text{ and } S^2 = \left[ \frac{n_\nu}{m^\nu} \right]^2 + \epsilon.$$

where for both  $\delta$  and  $\epsilon$  we have

$$\frac{\delta}{\epsilon} < \text{or} = \frac{1}{m^\nu} \left[ 2 \frac{n_\nu}{m^\nu} + \frac{1}{m^\nu} \right]$$

which is the difference between

$$\left[ \frac{n_v + 1}{m^v} \right]^2 \text{ and } \left[ \frac{n_v}{m^v} \right]^2.$$

But since  $P$  is a given number,  $(n_v/m^v)^2$ , being less than  $P$ , can never increase beyond limit; nor then can  $n_v/m^v$ . Thus if  $v$  is sufficiently increased the numbers  $\delta$  and  $\varepsilon$  are seen to be less than any arbitrarily small number. Every number which is *covered* by  $P$  is therefore also *covered* by  $S^2$ ; whence we obtain the equation  $P = S^2$ .

There exists a general number,  $x = S$ , for which  $x^2 = P$ .

16. The method of derivation of the general number exposed in the preceding pages follows in all essential features the Weierstrassian treatment; it was developed in detail by myself in the autumn of 1877 as introduction to a course of lectures on analytic functions. In this I made use of a report by E. Kossak (*Die Elemente der Arithmetik*, Berlin, 1872), which contains an attempted exposition of the Weierstrassian theory, but of which Weierstrass did not in any way approve. For example, on the fifth of April, 1895, he wrote to me, à propos of another matter: "You know how my introduction to the theory of analytic functions has been bungled by Herr Kossak and . . ." Kossak refers to a course of lectures given by Weierstrass in the winter semester of 1865-66; yet to anyone who is thoroughly acquainted with Weierstrass' manner of exposition and Weierstrass' methods, it might well be clear that he must have made known earlier in his Berlin lectures everything essential in his theory. In any case, he must have done so in the lectures on "Introduction to Analysis," which he gave in the winter of 1859-60 and in the summer of 1860; the former course having five lectures weekly, the latter four (and this latter without fee); and he had probably done so in his lectures of the summer of 1859 on "General Propositions Concerning the Representation of Analytic

Functions by Convergent Series" (public; one lecture weekly).<sup>14</sup>

The conception of his theory had certainly come to Weierstrass already in 1841 and 1842. As he himself insists, and as is also evident from the first four memoirs published by him (*Werke*, Bd. I), all the main features of his theory of functions were at this time already clear to him. The true basis of Weierstrass' theory of functions, the indispensable condition for arithmetization and for the rigorously abstract construction of mathematical analysis from the bottom upwards, is, however, and must be, the establishment of the two concepts of *general number* and *limiting-value*; and these concepts could not, before Weierstrass, be explained except by means of more or less hazy geometrical intuition.

Heine complains in his memoir, *Die Elemente der Funktionenlehre*, which appeared in 1872, and therefore in the same year as Kossak's publication, that Weierstrass' theorems, although proved by Weierstrass as "indispensable fundamental theorems," were nevertheless known only through his lectures and verbal communications and indirectly through lecture notes; and that their truth was, on this account, still doubted by many. This is also the reason why Heine, basing himself on verbal communications from Weierstrass himself, and with the active coöperation of Cantor, published his work. In it, however, on Cantor's advice, he has so far modified Weierstrass' fundamental definition as to start, not from the group  $((a))$ , but from an unlimited sequence of numbers, integral or fractional,  $u_1 < u_2 < u_3 < u_4 < \dots$ , such that if  $n$  be taken large enough, the difference between one number  $u_n$  and another  $u_{n+m}$  becomes less than any given but arbitrarily small number  $\delta$ —and that independently of  $m$ . He calls this sequence of numbers a *fundamental sequence*.

<sup>14</sup> *Verzeichnis der von Weierstrass an der Universität zu Berlin gehaltenen und angekündigten Vorlesungen*. *Werke*, Bd. II, pp. 355-6.

This "fundamental sequence" now defines a certain something which will be the new number; this becomes a symbol, a sign.<sup>15</sup> Heine also feels compelled to express himself thus: "In framing the definition I take the purely formal standpoint, *in that I call certain tangible signs numbers*, so that the existence of these numbers does not come into question."<sup>16</sup>

Thus the Cantor-Heine starting-point in reality becomes

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \sum_{v=1}^{v=n} a_v = \sum_{v=1}^{v=\infty} a_v;$$

this definition, as will appear later, is, in our exposition of the Weierstrassian theory as a whole, not given until after the concept of the general number with more than one fundamental unit has been introduced, and an elucidation of the essential properties of this number has been given.

The concept of limiting-value is, with Cantor-Heine, *postulated* at once at the beginning of the theory, and not, as with Weierstrass, *proved*. For the series

$$\sum_{v=1}^{v=\infty} a_v;$$

is, of course, nothing but a limiting-value of the numbers  $u_n$ , where  $n = 1, 2, 3, \dots$

At about the same time as Cantor-Heine, Méray<sup>17</sup> gave a theory of the general absolute number, which he defines in the same way as Cantor-Heine. Their "fundamental series" is called by Méray *variante*. That Méray, no more than do Cantor-Heine, regards as a true number the "incommensurable" number which he introduces, is apparent from his observation: "Telle est pour nous la nature des nombres incommensurables; ce sont des fictions permettant

<sup>15</sup> See *Journal für d. reine u. angew. Mathematik*. Bd. 74, pp. 172-188.

<sup>16</sup> *Loc. cit.*, p. 173.

<sup>17</sup> Méray, *Nouveau précis d'Analyse infinitésimale*. Paris, 1872.

d'énoncer d'une manière uniforme et plus pittoresque toutes les positions relatives aux invariantes convergentes."

In his time the same uncertainty was to be found concerning negative numbers, and shortly before our time concerning complex numbers. They were no true numbers, but a kind of symbols, on which substantially the same calculating operations could be performed as on true numbers.

The view adopted in the Weierstrassian theory is different from this. The assemblage of numbers which we have called a *general* absolute number is a creation which is given with and by means of the first conception of number and is arrived at by means of the capacity of thought to correlate with every number in the number-sequence, 1, 2, 3, . . . ,  $\nu$ , . . . a new number, and to regard the object represented by the correlated numbers as definite a real thing as is the unlimited sequence 1, 2, . . . ,  $\nu$ , . . .

Dedekind<sup>18</sup> has introduced the general number from a point of view different from that of Cantor-Heine-Méray. He writes:

"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."<sup>19</sup> (See p. 11 of the English translation.)

"I am not in the position to bring forward any proof of its correctness, nor is anyone in that position. The assumption of this property of the line is nothing other than an

<sup>18</sup> *Stetigkeit und irrationale Zahlen*. Braunschweig, 1872; 2nd ed., 1892. Cf. also, *Was sind und was sollen die Zahlen?* Braunschweig, 1888; English translation published by The Open Court Publishing Company.

<sup>19</sup> *Loc. cit.*, p. 18. "Zerfallen alle Punkte der Geraden in zwei Klassen von der Art, dass jeder Punkt der ersten Klasse links von jedem Punkt der zweiten Klasse liegt, so existiert ein und nur ein Punkt, welcher diese Einteilung aller Punkte in zwei Klassen, diese Zerschneidung der Geraden in zwei Stücke hervorbringt."

axiom, by which we first attribute continuity to the line—by which we think continuity into the line.<sup>20</sup>

From the geometrical point of view, Dedekind's position, that "the existence of the general number is nothing other than an axiom," can be maintained; but not from the arithmetic and conceptional point of view. However, Dedekind's geometrical definition can, as he has already himself insisted, without difficulty be transformed into one purely arithmetic.<sup>21</sup> This was done by Lipschitz in his *Lehrbuch der Analysis*,<sup>22</sup> which appeared five years later than *Stetigkeit und irrationale Zahlen*. If we follow Dedekind, we always, however his exposition may be varied, start from two "fundamental series," or "variantes," instead of from one, as we did with Cantor-Heine or Méray.

Dedekind's theory has, however, been much more frequently adopted in text-books than has any other. The explanation of this seems to be that it makes an appeal to geometrical intuition. There is imported into the very foundations of arithmetic a geometrical conception which is, and should be, foreign to it; and so one attains an apparently greater perspicuity, which the geometrical perception appears to bring with it.

G. MITTAG-LEFFLER.

DJURSHOLM.

(Authorized English translation by B. M. Wilson)

<sup>20</sup> "Ich bin ausser Stande, irgend einen Beweis für seine Richtigkeit beizubringen, und Niemand ist dazu im Stande. Die Annahme dieser Eigenschaft der Linie ist nichts als ein Axiom, durch welches wir erst der Linie ihre Stetigkeit zuerkennen, durch welches wir die Stetigkeit in die Linie hineindenken."

<sup>21</sup> *Was sind und was sollen die Zahlen?* Vorwort, Harzburg, 5 Oct., 1887. Braunschweig, 1887, 1893.

<sup>22</sup> Bd. I, Bonn, 1877.

## THE NEW PHENOMENOLOGY

TWO different attitudes are possible in regard to the world that surrounds an individual. The individual may, and generally does, take the world as "actually given" as something of which he himself is an active part. He lives in it, perceives its objects, hears its melodies, unveils its secrets. He acts, thinks, forms and unforms his beliefs; he creates new laws, and breaks the old ones; he loves and hates, votes and chooses. His science extends for him the field of his physical vision; his technique conquers for him new spheres, new worlds of action. In other words, the objects of the world appear to him in the capacity of service,—I mean service, not in the practical sense of utility, but in the sense of being a valid part of a valuable context. Everything is considered only insofar as it fits a context. Things, truths, theories are there for the individual to be believed or disbelieved; they exist or subsist for the sake of consequences, i. e., for the support of something else, as parts serving a system. They all have their dramatic careers within the general course of events, or general system of knowledge. They exist and subsist for the purpose of being listened to. That is what Husserl calls "*natürliche Einstellung*"—natural attitude—of the individual in regard to the world.

It is contrasted with the "phenomenological attitude" which is characterized by elimination of any service. We may still perceive things as real and existing but we shall

agree not to make any theoretical use of their existence, i. e., we shall not attempt to prove or disprove anything by the fact that they are real. It does not imply that we shall doubt their existence. We simply disregard the consequences of it, "take it into parenthesis," as Husserl says, "cut it off" from our judgment and remove its validity. We also may be conscious of truth, and precisely in the element of truth; but it shall not affect us by that side of its nature with which it performs functions in the system of knowledge,—it shall not be taken as a source of information. Considering any perception, judgment, or truth we shall not listen to what they actually claim and testify. We may watch the roles they play on the stage of knowledge; but we must do it as outsiders, not believing or disbelieving their testimony.<sup>1</sup>

Any thesis that we might have accepted in the natural attitude we may also retain and analyze from the phenomenological standpoint; but we must do that under the assumption that our "natural" beliefs are disregarded. "We are not giving up our natural attitude, and alter nothing in our beliefs, which are allowed to remain in themselves what they are. . . . And yet our thesis in regard to nature receives a modification: we let it rest in its own content, but we put it, so to speak, out of action, take it into parenthesis, and shunt it off. It is still there as the parenthesis in the brackets, or as the eliminated parts outside of the system to which they belonged. . . . We make of it no use,"<sup>2</sup> i. e., it is of no service. "I eliminate all sciences which are related to the natural world," says Husserl further, "although I do not intend to object against them, I make absolutely no use of their validity. Not a single proposition that belongs to the natural context, and is perfectly evident to me, is now admitted as valid or invalid; not one

<sup>1</sup> Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie* (1913), p. 187.

<sup>2</sup> Husserl, *Ideen*, p. 54.

is actually accepted; not one serves me as a foundation. . . . I may consider any proposition, but only after it is placed into parenthesis, i. e., only in the modified consciousness characterized by the elimination of judgment, that is, precisely, not in the manner it serves as a proposition within the science, not as a proposition that claims validity, which I may acknowledge or utilize."

The possibility of such a radical change of attitude is not confined to purely rational contents, such as truth, judgment, etc. The content of an artistic enjoyment or purely religious inspiration may similarly be approached from two different standpoints. We may be interested in theological dogmas, for instance, insofar as they support our religious beliefs and represent the system of faith actually needed among certain individuals to secure that type of life they believe to be the best; we may attack or defend their content; we may discuss their origin and explain their meaning. In other words we may take a stand within the net of faith. In this case our interest and attitude are those of a theologian, or student of religions. But we may, on the other hand, intentionally disregard the theological validity of dogmas, and take them merely as expressions of belief, as a peculiar attitude of consciousness to its objects which we call "faith." We pay no attention to their content, taking it into parenthesis, and study only the mode of its peculiar givenness. We take them as they actually give themselves in the *element* of faith, but we utterly disregard what they practically accomplish in the *system* of faith. In such case we proceed as phenomenologists. Thus phenomenology defines itself as a *study of objects in their relation to consciousness in a state of supreme impartiality, when the face-value and systematic significance of the objects concerned does not come into consideration*. This fundamental impartiality, or *abstinence* from the actual validity of objects, is something more than a mere "state of consciousness,"

or arbitrary invention of an idle mind. It implies a method, and proclaims a new intellectual policy in regard to the whole world, and even more. Any object whatsoever—it may be real or unreal, logical, alogical, or even illogical—may be phenomenologically approached, or *purified*, i. e., deprived of its natural or systematic connections. Such a theoretically “disconnected” object Husserl calls “Phenomenon,” and the methodical abstinence by which phenomena are thus obtained “phenomenological reduction.”

Thus phenomenology is primarily characterized, not by any particular choice of objects, but rather by the method of approach. The material of phenomenological inquiry is the same as in other sciences,—the world and its innumerable phenomena. But the method of approach is fundamentally different. Phenomenologist does not proceed metaphysically: he does not invent any super-world for his own intellectual amusement in order to obtain an object for his study that could not be studied by any other branch of science. He attempts to describe the same world which is familiar to everybody; but he does it in the attitude of phenomenological impartiality; he *suspends* the judgment in regard to any and all phenomena as far as they are parts of a larger system; he attempts to isolate any given object from the context in which it is actually engaged, and contemplates it in its own purified *essence*.

But—one may ask—why should the objects be isolated? Why should the judgment be suspended? In other words, what is the purpose of the phenomenological attitude? Since Hegel it became a truism that the context, i. e., systematic connections in which a given object is engaged, has a very effective influence upon the content or notion that we form about the object. This influence, as valuable as it is, may under circumstances be exceedingly harmful and misleading. Do not even the greatest thinkers often lose sight of their original issue in view of certain systematic

relations which make them answer questions they have never asked, and indefinitely postpone the solution of problems they have started with? Do not epistemologists, for instance, often forget the specific content of their epistemological problem under the influence of psychological theories (which in themselves are valuable and true)? Are not we, generally speaking, too prone under the pressure of systematic necessity to substitute one content for another? We often hear the physicist maintain that red color, for instance, does not in reality exist, and must be physically regarded as a certain frequency of ether vibrations. Some thinkers go even so far as to maintain complete reducibility of the phenomenon to physical causes; what we call "red"—they say—is nothing else but a peculiar form of energy discharged by the brain cells! And when the psychologists attempt to correct this materialistic conception, by pointing out that "redness" itself is real as a "sensation," they commit a similar error; for they interchange certain theoretical content, called "sensation," with the genuine phenomenon of "red" as such, which in itself is as strange to sensations as it is to vibrations.

Modern science and philosophy display a regular mania for similar reductions. We are trying to "reduce" nearly everything. Social life is "reduced" to either economic or psychological "factors"; life in general is reduced to "purely mechanical causes"; sound is reduced to vibrations, and vibrations to the principle of conservation of energy; phenomena are reduced to laws, and laws to principles. Repeated attempts are being made in philosophy to reduce the whole world to one substance, one God, or one fundamental principle, such as  $I = I$ . A thing or proposition is not considered sufficiently clear until it is reduced to something else. The nature of "explanation" consists in reduction.

Many of those reductions are valuable, indeed. But often they merely obscure the issue, and substitute the inventions and traditions of our mind for the actually given phenomena. "A definite shade of red"—says one of the leading phenomenologists of Husserl's school, Max Scheler—"may be determined in many different ways. For example, as the color that is enunciated by the word "red" (color itself being already a substitution, a reduction); as the color of this thing or this particular surface; as the color that "I see"; as the color of this particular number and length of vibrations. It appears here as an X of an equation. The phenomenological experience alone can give us the "red" itself, in which the totality of those determinations, and signs, and symbols find their ultimate fulfillment. It transforms the X into a fact of intuition."<sup>3</sup> This fact, i. e., the phenomenon of "red" as such, seems to have no place within any of these *contexts*; it can not be reduced to either psychological or physical elements—and in our days it was once more emphatically pronounced an "illusion."<sup>4</sup> Thus context kills the phenomenon. Our conception as to how "red" should exist, i. e., existential connections of "red," or—popularly expressed—our theories in regard to "red" as reality, remove the phenomenon itself from the field of our intellectual vision, and leave us in position of a man who in view of the trees does not see the forest. For this reason it is important to suspend our judgment in regard to any such "reductions," or—in other words—to reduce our judgment to pure fact. In this case our reduction is precisely the reverse of an explanation; it is the process of clearing up facts, i. e., finding the genuine—"phenomenological"—content of objects. Our objects are covered with "theories" to such an extent that to recover them is by no means an easy task. We are so accustomed to regard

<sup>3</sup> Max Scheler, *Der Formalismus in der Ethik und die materiale Wertethik*, p. 45.

<sup>4</sup> Cf. Hermann Cohen, *Logik der reinen Erkenntnis*.

"red," for instance, as a sensation or vibration that any statement in regard to its "independent" value meets a violent opposition on our part. We would rather declare it an illusion than consent to regard it as an independent entity. The important fact is, however, that being even an illusion it cannot cease to be "red." We cannot get rid of its "essence" even by pronouncing it illusory.

But—it may be objected—granted that "red" has a certain phenomenological essence that lies outside of any physical or psychological context, what is that we gain by taking it into consideration? What can we study about "red" as such, independent of physical or psychic connections in which it appears as real? I must admit, indeed, that there is not much to study about it except the method and idea of the phenomenological attitude (from which the neo-realistic position, for instance, follows as a mere corollary, —a result which apart of any other consequences must be considered as worth something!). The phenomenological "independence" of the secondary qualities makes the radical change of the attitude clear and comprehensive. And phenomenologists constantly refer to colors and tones for the sake of illustration. But the value of phenomenological attitude is scarcely indicated, by no mean exhausted by pointing out the phenomenological independence of secondary qualities. The method of phenomenological reduction brings more definite and constructive results, if applied to other phenomena such as "knowledge," "imagination," "value," "beauty," etc. The phenomena of "truth" and "meaning," for example, need even in our days a great deal of phenomenological purification. They are too often mistaken for psychological realities. Do not some prominent thinkers of today regard "truth" as a result of our mental

organization,<sup>5</sup> perhaps, even a by-product of our biological structure?<sup>6</sup> Is not it rather modern to maintain that there are no eternal truths, and that all truth is made to satisfy our biological or social needs, to meet certain difficulties? In direct opposition to these modern misinterpretations of "truth" as a kind of mental reality, fact, or function of mind, Husserl works out his phenomenological conception of truth, as a phenomenon *sui generis* which, in its essence, is entirely independent of psychic connections. In his "Logical Studies" he proceeds to demonstrate that truth cannot be considered as reality. It has no existence.<sup>7</sup> A proposition does not start being true when we first learn it, and it does not cease to be true after we completely forget all about it, or after even the whole human race entirely disappears from earth.<sup>8</sup> Here again, as in the case of "red," by disregarding the existential connections of "truth," i. e., our theories in regard to the mode of its existence, we obtain the pure phenomenon of truth in the form in which it is actually given in the act of knowledge or evidence. For in the act of knowledge truth is originally not intended as a mental reality; this latter is superimposed upon it by theoretical considerations of an entirely heterogeneous nature. Truth does not appear as a mental reality to the knowing individual, but merely to the theoretical epistemologist. When we say that  $2 + 2 = 4$ , we originally do not have in mind any combination of mental processes but merely that of numbers. That numbers may be real as mental states does not change anything in our mathematical intention; if some one will definitely prove that they are *not* mental states, it will have no effect upon the mathematical science. The important feature of this

<sup>5</sup> Cf., for instance, G. Heimans, *Gesetze und Elemente des wissenschaftlichen Denkens*, pp. 38, 64-65, 90, 181-189, 224, 242, etc. Also Sigwart, *Logik*, Vol. 2, p. 66, 40-41, 92. For a more detailed analysis of the psychologistic viewpoint see H. Lanz, *Das Problem der Gegenständlichkeit in der modernen Logik*.

<sup>6</sup> Comp. Mach, *Erkenntnis und Irrtum*.

<sup>7</sup> Husserl, *Logische Untersuchungen*, Vol. I, p. 119; Vol. 2, pp. 590-595.

<sup>8</sup> *Logische Untersuchungen*, Vol. 1, pp. 100, 117, 131, 184, etc.

argument consists in clearing up our own intellectual intentions. Husserl does not say that the above expression  $2 + 2 = 4$  subsists, like a Platonic idea, somewhere outside of our mental processes; but he maintains that when we add two and two we do not intend to add two ideas; what we have in mind is not a psychological, but purely arithmetical relation. And only in this sense the truth expressing this relation is different from psychological reality. Thus phenomenological attitude helps us to grasp truth as a phenomenon *sui generis*—a task that opens a new field in which there is still much to be accomplished.

Phenomenologically speaking every object is totally different from the state of consciousness in which it appears as given. We often say that certain things exist merely in our imagination. In reality, however, we can never find any of those imaginary objects within the real psychological content of imagination as such. Suppose I imagine the god Jupiter. By virtue of my imagination Jupiter is endowed with various mythological qualities, such as omnipotence, physical force, certain face, beard, etc., which by no means can be attributed to my imagination. He was a married man,—a situation that none of my ideas can be possibly imagined to endure. One may analyze, says Husserl, his idea of Jupiter down to the smallest detail which his own introspection or modern psychological methods will allow to get hold of, he will never be able to find Jupiter himself within the constituents of his idea. For the ideas are real occurrences, and Jupiter is not a real occurrence: he does not exist anywhere.<sup>9</sup> The same argument applies to any idea, any proposition, belief, or perception, in brief, to any act of consciousness whatsoever, provided that it "has something in mind," or "intends" something. The object of intention is phenomenologically different from the act of intention; different, not as an independent

<sup>9</sup> Husserl, *Logische Untersuchungen*, Vol. 2, p. 352.

reality, but merely as a *different center of possible predication*. Propositions which are valid in regard to an object are generally invalid in regard to the act, with the exception of those cases when the object is another psychological act.

The critical dogma of "Identity" is thus abandoned. In the light of the theory of intention objectivity regains its scholastic "independence." It was a veritable resurrection from the dead. Kant buried the objects in the depths of consciousness. Husserl again extracted them from the grave, gave them a new life, and restored them to all the honors and titles of *ens intentionale*.<sup>10</sup> But the hundred years they have spent in the grave left a profound change in their nature that no restoration could possibly erase. The very heart of objectivity was left infected with transcendental problems and ineradicable craving for a priori. Mediaeval essences reappeared on the philosophical horizon, and again assumed the leading role in the life and organization of thinking. But they changed the modus of their existence, and became more "epistemologized" and more restricted to their own "intentions." In brief, the objects regained their independence, with the essential limitation however that they should be taken as the intention actually gives them. If they are so taken, they are bound to appear in their phenomenological essence, i. e., apart of their factual existence, and strictly a priori. For to carry out the "phenomenological reduction," i. e., to isolate an object from its existential or systematic connections, is equivalent to considering it as it is originally given, without the distorting influences of "theory." Thus considered every object is bound to reveal what it is in itself, "red as

<sup>10</sup> The immediate source of Husserl's intentionalistic inspirations lies, not in the scholars of the Middle Ages, but in Brentano's Psychology. Says Brentano: "Jedes psychische Phänomen wird dadurch charakterisiert, was von den Scholastikern die intentionale Inexistenz des Gegenstandes genannt wurde, und was we jetzt das Gerichtetsein auf das Object nennen würden." *Psychologie*, 8, 115.

such," "truth as such," etc. And that means to reveal its *ens intentionale*. Thus Meinong's *Gegenstandstheorie* appears as a mere corollary of the phenomenological principle.

The methodological importance of *ens intentionale* may be shown by the following illustration borrowed from phenomenological aesthetics. Since Aristotle it became a popular view that artistic enjoyment consists of the feeling of relief which is caused by a work of art—a relief from the burdens and sorrows of life, from the passions and worries of action. That is what Aristotle called "*καθαρσις*," or artistic purification. To see the essence of artistic enjoyment in such purification is, from the phenomenological standpoint, altogether wrong. For such purification, even if it does exist, is merely a consequence of artistic experience, a by-product which is in itself desirable, but does not constitute the phenomenon of artistic enjoyment as such. For the latter is an act of enjoying the work of art, and not the relief from passions. To substitute one for another is again a result of our mania for reductions. Going to a theatre I perhaps enjoy the relaxation from my daily work; I am also glad to have an opportunity of forgetting my troubles. But this kind of enjoyment is not that which I derive from watching the drama played on the stage: I should be able to enjoy Shakespeare even if I had nothing to forget. My relaxation and recreation are certainly enjoyable, but it is enjoyment of relaxation, not of art. Thus clearing up our actual "intentions" we avoid confusion which may, otherwise, prove fatal, not merely to our aesthetical theories, but to our artistic tastes and practices as well.<sup>11</sup>

Together with Schelling, and furthermore with Plato, Husserl believes that the only means of reaching *ens intentionale* lies in pure intuition.<sup>12</sup> He is inclined to

<sup>11</sup> Moritz Geiger, *Beiträge zur Phänomenologie des ästhetischen Genusses*, p. 592-594.

<sup>12</sup> Husserl, *Ideen*, p. 113.

regard intuition as the only solution of epistemological problem.<sup>13</sup> "Phenomenological intuition" is experience in the sense that we arrive at it through the immediate contact with its data, and not by means of reasoning or speculation. Reasoning does not produce its content. Since the content is admittedly different from the process by which we are aware of it, no reasoning will help us to create it. It has to be given to us, and unless it is given we cannot be aware of it. In respect to awareness an abstract concept, such as number five, is in the same position as any sense quality: the relation of consciousness to its object is in both cases fundamentally the same; number five has to be "present" to consciousness in precisely the same sense in which "red" is present. Thus intellect is itself intuition. That does not mean, however, that phenomenology is an inductive science. In order to know that "two straight lines can cut each other in one point only," or that "in the pyramid of colors orange occupies a place between red and yellow," we do not have to refer to a number of similar cases. Both propositions, although they are statements of fact, are not derived by induction as they are not generalizations from individual cases. One single look at the pyramid of colors is sufficient to convince anyone that the said relation between "orange," "red," and "yellow" is valid for all cases, that it must be valid. We see, in other words, that it is an essential relation that cannot be changed, and cannot be even imagined to be formed in any other way. An appeal to other cases is of no assistance; for the evidence is complete on the basis of one single case.<sup>14</sup> In this sense the relation is both "given" and "a priori." Phenomenologists call a priori all those meanings and propositions which are based on evidence furnished by intuition. To reach a statement that would be in this sense a priori it is necessary to surrender all

<sup>13</sup> *Ib.*, p. 204-205.

<sup>14</sup> M. Geiger, *Beiträge zur Phänomenologie des ästhetischen Genusses*, p. 571.

beliefs and theories, to abstain from any kind of "Setzung," such as "real" or "unreal," "true" or "illusory," and to give oneself up to the content as intended. A content which is independent of the contrast of "truth" and "illusion," etc., which is, in other words, equally necessary in the world of truth as well as in that of a fairy tale, is properly called by the phenomenologists "essence." Thus the essence of life must be given even if we are under illusion that a certain object is alive; the essence of "law" must be present in the constitution of the United States as well as in that of a Lilliput kingdom. Essence in this sense is neither a universal nor individual term. The essence "red," for instance, is contained in the general concept of red, as well as in every concretely perceived shade of red color.<sup>15</sup>

Moreover, essences in the above sense are not necessarily rational. There are perhaps innumerable forms of intention or modes of awareness (*Bewusstseinweisen*) which all have their objective counterparts. Faith, doubt, imagination, inquiry, enjoyment, wish, etc., are names referring to different modes of awareness, each having its own specific nature, or general essence, which transmits itself to its objects. An object of faith has its own peculiar flavor that makes it "essentially" different from any object of reason; in spite of the fact that both may coincide in certain points they are fundamentally different precisely insofar as one belongs to the realm of faith, the other to that of intellect and reason. The will-consciousness similarly results in a peculiar kind of objective structure which we generally call "values," and which have their own relations and connections which are distinctly different from those of "truth" or "reality."<sup>16</sup> It is futile to rationalize those relations, for they are non-intellectual. They can be adequately grasped only through "feeling," and in the

<sup>15</sup> M. Scheler, *Der Formalismus in der Ethik und die materiale Wertethik*, p. 43.

<sup>16</sup> M. Scheler, *Der Formalismus in der Ethik*, pp. 10, 15, 59, 79, 133, 260-272.

midst of the difficulties and entanglements of real action. A value is neither "thing," nor "truth," and its relation to other values is neither physical nor logical. Our contact with values, therefore is of a different order. Blaise Pascal called it "logique du coeur." M. Scheler calls it Ethos. It is one of the alogical forms of objectivity.

In this sense there are as many types of objectivity as there are types of consciousness. Each kind of "intention" is, so to speak, supplied with its own material, which forms a peculiar world or ontological "district" in itself, with its characteristic district-essence and district-categories.<sup>17</sup> The world of faith has a specific type of objectivity that is different from that of emotion; the world of things is, again, essentially different from that of "goods," or "values." Each type implies, according to Husserl, a twofold relation: 1. It is necessarily related to the reality of psychic events, to my ego, or in other words, to those "parts and elements" which can be actually found within the act itself, as belonging to the object of inner sense. This subjective, although not necessarily psychological, aspect of consciousness Husserl calls "noesis." In this term the subjective element (the mode of consciousness qua consciousness) receives its formal recognition in the system of phenomenological philosophy. 2. On the other hand, each particular mode or type of consciousness, such as faith, imagination, reflection, etc., is an *actus intentionalis* that points out to something beyond. The objective character of that something changes in accordance with the essential character of noesis. To characterize a given content as an object is utterly insufficient; we have to add the specific "how" of its givenness (*das Wie seiner Gegebenheitsweise*). In other words, each type of consciousness by virtue of its own intention is intuitively projected into the sphere of objects among which it then appears as an objective

<sup>17</sup> Husserl, *Ideen*, p. 19.

essence. Those projections Husserl calls "noemata."<sup>18</sup> Thus perception has its noema, that is, perceptum as such;<sup>19</sup> every question has its noema, that is, its meaning as a question, precisely as it is intended by the inquiring mind;<sup>20</sup> in judging about something we place ourselves in contact with that peculiar world of neutral entities which are generally called logical contents, or "truths" (thus neutral entity appears as a specific case of noemata). Generally speaking, "to the manifold data of real noetic content everywhere corresponds a manifold of data in a noematic content." Husserl insists that these data should be described precisely in terms of that "mode" or type of awareness in which they are originally given. The noemata that correspond to our wishes or actions are fundamentally, "essentially" different from those which constitute the world of thought or sensation; they have their own specific character that cannot be under any circumstances reduced to intellectual elements. That settles the question of ethical intellectualism as well as sensualism. A value can not be logically demonstrated; for its "essence" is alogical. Kant's categorical imperative is an intellectual delusion—a mirage of will in the deserts of formal reasoning. But to think of value in terms of psychological analysis, as an anticipation of pleasure, is equally impossible. The stronger is our wish the more we forget about its psychological constituents, and lose ourselves completely in "plans" and "schemes," in "ends" and "means." That, partially perhaps, accounts for the fact that the greatest geniuses of will in history were inclined to regard themselves as tools of either divine power (Cromwell and his circle), or fate (Napoleon, Wallenstein); they were lacking the consciousness of their ego as the immediate source of their will power. The only possible way of arriving at positive and constructive ethics

<sup>18</sup> Husserl, *Ideen*, p. 179-199.

<sup>19</sup> *Ib.*, p. 182-183.

<sup>20</sup> *Ib.*, p. 197.

lies, therefore, in recognizing the original content of ethical values—an independent order or logic of heart.<sup>21</sup>

Noeses and noemata together constitute the realm of absolute consciousness. They are absolute "facts" that no scepticism can possibly eliminate, a realm of absolute Being that "nulla re ad existendum indiget."<sup>22</sup> Whatever is, or may ever be known, is based on such facts. In this sense neo-phenomenology is restoration of empiricism.<sup>23</sup> "The facts and facts alone, and not the constructions of an arbitrary "reason," form the foundation of phenomenology." But its facts are different from those which form the basis of natural science. We find here, in the mathematical sense of the word, a generalization of the concept of fact. A phenomenologist would never attempt to reduce an "essence" to "facts"; for, from his standpoint, the essence itself is a fact. And, on the contrary, what is commonly called fact (in scientific sense) is merely a special case of essence. Neo-phenomenology is neo-empiricism. It is a continuation of Hegel's daring experiment to bring essence and fact in one. Husserl and Hegel, however different they are in the outlook and outcome of their doctrines, both agree in the effort of making philosophy a continuous stream of experience. Nothing is excluded from experience. Truth and falsehood, real and unreal, essence and fact, individual and universal—all are equally justified as "contents" or "stages" of experience in the purified consciousness of phenomenological impartiality. The phenomenologist, like the pragmatist, can never be bothered with any religious or moral doubts; for he is above (perhaps below?) all problems. The actuality of a problem never interests him. His interests are immense; but it is an immensity of surface. He acknowledges everything, but it is the acknowledgement of a formal receipt. This phenomeno-

<sup>21</sup> M. Scheler, *Der Formalismus in der Ethik*, p. 59.

<sup>22</sup> Husserl, *Ideen*, p. 91-92.

<sup>23</sup> M. Scheler, *Der Formalismus in der Ethik*, p. 46.

logical universality of interests, this cold-blooded cosmic anatomy deprived of amor intellectualis in Spinoza's sense and united with the fundamental impartiality of judgment, is the common feature of Husserl's and Hegel's philosophies. Both are partisans of impartiality.

HENRY LANZ.

STANFORD UNIVERSITY, CALIFORNIA.

## REASON AND FREEDOM

THERE are two very old problems concerning the psychological basis of ethics, which have during the last twenty years of psychological progress aroused fresh interest. The one problem is that of the freedom of the will. The other problem, distinct from the first but by no means unconnected with it is the problem, as old as Aristotle and Plato, as to whether reason has any real power in itself to determine choice. As regards the first problem there is a very strong bias among the "instinct" psychologists and the psycho-pathologists to assume complete determinism and deny freedom.<sup>1</sup> As regards the second there is a still more powerful tendency to regard reason in itself as impotent really to determine choice. It follows, and expresses in its new terminology, Aristotle's criticism of Plato, that in moral choice there must be desire as well as reasonable or intellectual knowledge. But it often goes further and asserts that all desires are the outcome of innate tendencies or instincts, and that reason can only discover means to instinctive ends. I do not wish to offer a criticism of "instinct" psychology, but I shall try to show that "reason" is more important than is often supposed, and that its possession may even be the real condition of freedom.

To both problems mentioned above Doctor McDougall, who may be described as one of the "fathers" of the mod-

<sup>1</sup> Among the pathologists this attitude is not without exception, as, e. g., Dr. William Brown, who follows Jung in rejecting determinism.

ern movement, has given his contribution. Of determinism he spoke in the *Social Psychology* of 1908, as "this dark shadow on human life."<sup>2</sup> But he expressed himself willing "to believe in a little dose of free-will if the conception can be made intelligible." It is interesting to notice that McDougall now, in contrast with the Freudians, repudiates determinism on the ground that creation is a fact.<sup>3</sup> He now says "the belief in 'strict determinism' on the part of a man who actively pursues his goals, and puts forth strenuous efforts is . . . merely a symptom of mental disorder of so mild a nature that there may be good hope of his recovery."

On the other point, as to the adequacy of reason to determine choice, McDougall expresses himself in unambiguous language. "Unless," he says,<sup>4</sup> "a man really hungers for righteousness, already desires to do whatever is right, to be whatever is virtuous, unless, that is, he possesses the moral sentiments and moral character, reason cannot impel him to do right or to desire it. To create desire is a task beyond its competence; it can only direct preexisting tendencies towards their appropriate objects." And again,<sup>5</sup> "Shall we be content to say, with Plato and some modern moralists, that Divine Reason sits in the head, controlling fierce passions that reside in the belly, as a charioteer controls with whip and rein a team of savage steeds? Hardly! Reasoning plays an important part, as we have recognized, in refining and harmonizing the moral sentiments. And reasoning may help us to acquire a moral creed,"<sup>6</sup> such as that happiness or perfection is the good, or it may enable us to discover the best means to the end. But "Reason" "is not a conative energy that may be thrown on this side or that, in our moral conflicts. Reason is not that X of which

<sup>2</sup> *Social Psychology*, p. 235.

<sup>3</sup> *An Outline of Psychology*, pp. 447-448.

<sup>4</sup> *Social Psychology*, p. 379.

<sup>5</sup> *Outline of Psychology*, p. 439.

<sup>6</sup> *Outline of Psychology*, p. 439.

we are in search; though it plays an important part in bringing that X to bear."† The X, for McDougall, is "an impulse awakened within the sentiment of self-regard." These views may be admitted to be in large part true. Reason is certainly not the "conative energy" which some rival-ists have supposed. But it may have much more power to control the "fierce passions that reside in the belly" than is here allowed. Before discussing this, let us digress for a moment to define four senses in which the term "freedom" may be used. For the problems of the place of reason, and of human freedom, must, although distinct as we said, be discussed together here. Reasonable choice, we shall contend, is in *some* sense free choice.

Of the four senses in which "freedom" may be used, there is, (1) the sense in which it means unmotivated choice. This is usually called "indeterminism" and leaves the nature of choice a mystery. (2) There is the freedom which reason is supposed to possess of choosing between "higher" or "lower" courses of action. This has been called "neutral" freedom. (3) There is the "higher" freedom attained only when we make a particular virtuous, rather than a particular vicious choice. This is the freedom of which St. Paul and Kant speak. St. Paul says, "the law of the spirit of life in Christ Jesus hath made me free from the law of sin and death."‡ This is the "*non posse peccare*" of the mediaeval theologians to which Sidgwick refers.§ (4) There is a fourth sense possible; a freedom only to be realized fully in the ideal, of the man who is habitually virtuous, and who is therefore permanently released from the temptation of the lower man.

With (1) we shall not be concerned. It is of no importance today. It should be noted that (2), (3), and (4) are continuous stages on the plane of rational conduct. We

† *Outline of Psychology*, p. 440.

‡ *Romans* VIII, 2.

§ *History of Ethics*, ed. V., p. 261.

shall discuss mainly the second sense (2), as it is the crucial case, and shall afterwards pass to discuss (3). In the meantime, let us examine shortly some of the implications of the possession of reason, and its bearing upon any freedom which rational minds may be said to possess.

By the term "reason" I mean here the power which mind possesses of analyzing and synthesizing into ideal wholes the materials presented in experience. The perfection of reasoning as we know it is found in the proofs of natural science and in mathematics. But we shall find that it is rather in its implications than in its actual processes of induction and deduction, that the power of reasoning is important for moral choice. Reasoning implies a certain ability to stand off from objects both of existence and value, and to contemplate them more or less as they are. This is in marked contrast to the instinctive attitude, where objects are considered mainly in terms of our conative relation to them. Let us examine the power of reason to contemplate objects more or less objectively.

It is usually admitted that good thinking and good reasoning imply at least some *freedom* from prejudice. What this means is not as a rule considered, but it is true that, however we may be impelled to think, by curiosity, acquisitiveness, etc., we are, when we reach the thinking level, somehow and to some extent "free." Reflection is a weighing of pros and cons in which, if we are honest, impartiality is the ideal. We definitely set out to be free from bias; that is, we agree to accept, not merely subjective impulses or whims, but an objective (e. g., a consistency) test of truth. If the value of truth as such is known and felt by us to be an intrinsic value, if it is felt to be worth while to discover truth for its own sake, we can, although it is a somewhat rare quality, assume an attitude of impartiality. The good scientist takes account of all relevant facts, and the good philosopher should be prepared to try to take into

account as far as possible all facts, all experiences, all values. However he may be instinctively biased towards one view or another, he can by an effort, through contact with other minds, shake much of this from him. It is a difficult thing to do, but, as Plato says, "the cowardly and mean nature has no part in true philosophy." The philosopher is able, by means of his feeling for truth, to achieve things which to the man in the street seem "against nature." In civilized society the case of the impartial judge in a court of law is perhaps the most familiar.

I call the process of reflection "free" because it is a standing off from, and a real facing of alternatives. We must not, of course, suppose that the stage at which we reflect is not conditioned by evolution. The man who reflects truly when, e. g., he is faced with alternative courses of action, does not reach the moment of reflection undetermined by all conditions of heredity and environment, and of course his conclusions themselves are, if he is honest, determined by the desire for truth. But at the moment of true reflection to which he is urged, no doubt by evolution, training and habit, a man is able, as it were, to step out of the prison of immediate determination by immediate impulses, and to contemplate as possible other courses of action. That he can thus contemplate, can thus weigh possibilities, alternatives, is the only point which I am attempting to establish at the moment. I am now concerned with the weighing of alternatives rather than with the meaning of choice of one course as opposed to another. It is of vital importance for the later theory to understand it.

One of the most obvious characteristics of reflection is the *delay* which it involves. It is a weighing and a measuring and a selection as between immediate alternatives which present themselves, in the light of some more remote general principle. The particular alternatives are valued in the light of that more general value. Whether the

reflection be on a theoretical or practical question, the principle involved is essentially the same. Where my aim is theoretical knowledge, I test hypotheses by their consistency with known facts. My general principle is consistency (I here assume), and that is further a symbol for my idea of truth, which is thus the dominating value of all proper theorizing. In reflecting on affairs which have practical results (and it is hard to name any theory which in the long run has not such results with their concurrent temptations to bias), we may be biased by our preferences and select as dominating values, not the true ultimate good, but the good which seems most pleasant or convenient at the moment. Still, from the most theoretical of reflections with its ideal of pure truth, to the most practically biased, there is the element of delay. We are not inevitably impelled to follow one impulse or to adopt as ours one alternative, we choose one or the other because as an alternative it seems best to fit the end. When we represent to ourselves an object, we distinguish ourselves from that object and from the values whereby we estimate objects: that representation of self-distinction and estimation involves pause. This principle of pause, as we shall see shortly, contains in it the essential explanation of all voluntary behavior.

At the cruder stages of the moral life the term "pause" with its temporal connotation, has a very special significance. The animal, or the man who lives like an animal, gives no pause before action, but yields to immediate impulse. It is possible to construct a low sort of ethical hedonism based upon this idea of life, with "seek the immediate pleasure" as its motto (although the fact that immediate pleasure-seeking is made a principle raises it above animal level). The stage higher than this is that of, e. g., the Epicureans, who, still believing pleasure to be the highest good, yet advised the abandonment of the nearer, lesser

pleasure for the sake of the greater but more remote.<sup>10</sup> The highest stage would be that of the man who sought, not the greatest pleasure, whether remote or near, but the realization of a value, some intrinsic, objective value, desired for its own sake and not regarded as good simply because of its pleasantness to the desiring subject. This ideal would doubtless satisfy the man, but mere self-satisfaction would not be his end. Note the progression from the temporal life of the individual, to the life determined by a more abstract idea of an over-individual value. First comes the non-moral, or nearly non-moral, "living from moment to moment" of the animal or lower man. Next, the expectation of pleasures remoter from the immediate present, expectation based upon remembrance of the past, stretching over a wider area of human experience. Lastly, the life not based upon pleasures in time (whether present or future) at all, but upon intrinsic values to be sought and realized at all costs, even though in the act the individual subject should pass away altogether. Reflection is present in all these stages to some extent, though very little in the first, but the pause which it involves conditions vastly different qualities of choice in the different cases. The values vary from that of a mess of pottage to a birthright, from the pleasure of thirty shekels of silver to the value of fidelity.

The development, then, of the power of reasoning reflection makes possible the revelation to our mental eye of an ever increasingly wide range of both quantity and quality of choices from which we may select. The increase of range further means development of the power of looking at things in the large, of seeing them in true perspective.

It is this stage of reflection in the realm of moral life which is the stage of true "neutral" freedom. If there is no such thing as freedom of theoretical thinking, of objec-

<sup>10</sup> That this is a "pleasure of the mind" and not a sensuous one, does not affect the argument.

tive, unbiassed judgment (so far as necessary limitations of all knowledge admit), then all science, all theory, becomes a farce, and not less so the theory that so judges it to be farcical. We should be immersed, if this were so, in a bog of Protagorean relativity, and no one who is serious ever quite believes that, as he shows by his seriousness.

Moral judging, of which we have given instance, is just a form of theoretical thinking where the thought is turned upon objects of possible choice. It may be more difficult to be unbiassed and "objective" in this sphere, because we usually have an interest one way or the other, but it is not wholly impossible, as I have tried to show. Just so far as it is possible, just thus far is a man "neutrally" free.

But in saying this we are of course very far indeed from having solved the question of freedom of choice, for *knowing* the alternatives, and *knowing* which alternative is on the whole objectively the better, is not the same as *choosing* it. The act of choice goes a step further, it is self-identification with one of the alternatives. And this self-identification is always determined by something, whether it be the desire of the ideal or the desire of the more immediate object. It is therefore important to get a clear idea as to the relation of reflection to choice, as to what part of choice is free, and what is determined.

The matter can, I think, best be stated as follows: Evolution, heredity, environment, education, etc., have elevated us to a condition in which we are able to reflect: reflection, though conditioned by instinctive propensities, such as curiosity, is yet free from our own impulses in the sense that we are able to step back, as it were, to contemplate them and the ends they subserve, and, further, to value various alternatives by reference to some wider and more ultimate end. Evolution brings us to a level where we are not only able to contemplate our own purposes more or less objectively, but to contemplate for that very reason a wider

sphere of possible choice. So far, then, we can say that we are determined by the past, but that we are determined into a state of freedom where it is possible to explore mentally many alternatives. That reflection is determined by some interest in truth does not imply that one decision is determined rather than another. For the very bias towards truth *reveals* the alternatives, the honest weighing of which requires un-bias. It is however only whilst we are reflecting, only whilst we are in the process of choosing, only while we pause, that we can actually be said to be mentally free; before us then are a number of varying possibilities, some more worthy, some less. But so soon even as very reflection decides that one alternative is preferable to another, just so soon do we begin once more to be determined. The truth-value which is the very determinant of the *freedom* of reflective choosing, may enable us (in perfectly honest choosing) to know the alternative which is best in the given circumstances. So soon as this is actually preferred we can be said to be determined by it, though by a previously "free" process. If I let immediate bias creep in, then subjective interest prevails, and it is that which leads me to select A rather than B. If we may express it epigrammatically, choosing is free, but choice is determined. Choosing is freedom from impulse and the revelation of a wide range of possibilities; and when we freely choose, we freely *choose* to be *determined* by one end or another.

As to whether we allow immediate impulse, or an ideal, to determine us will depend, of course, very largely upon our heredity, environment and education, particularly the two latter. It will depend upon the kind of sentiments we have formed, e. g., upon whether we believe in more immediate satisfactions than in ideals. Only I would remark in passing that, important as those sentiments are, a single strongly-felt experience of an ideal value, which might

occur quite by chance, might be sufficient, either directly or through vivid recollection, to determine a course of action. But leaving this, we may say generally that, since reflection is so all-important, the value of a strong sentiment for truth cannot be over-estimated when discussing the moral life which is called upon to make new choices and is not merely conventional. Only by a development of this strong sentiment can we bring ourselves to that attitude of objective impartiality which enables us to value possibilities in the fairest way, and so mould our actions and lives upon the deepest law of things. In theoretical thinking, truth itself is the end. In practical thinking with a view to choice, truth is the end in the first place, but is in the second place a means to the realization of moral values which are different from truth.

It is perhaps worth while to notice McDougall's solution of James' well-known problem of will:

Ideal impulse  $+ x >$  instinctive propensity.

McDougall, as we saw, says that the  $x$  is an impulse awakened within the sentiment of self-regard. We may criticize this point and say that the whole *ground* of true regardfulness of self must be the worth of an *ideal* adopted by the whole self. If this is so, then James' formulation will be unnecessarily paradoxical, as indeed I think it is.<sup>11</sup> There will not be added an  $x$  to an Ideal impulse. The Ideal impulse, if it wins, is not weaker, but stronger than the instinctive propensity. It is not as a matter of fact true as a general statement to say with James and McDougall that an ideal impulse is *necessarily* weaker than an instinctive impulse. A strong feeling for something felt

<sup>11</sup> As a matter of fact, James desires this formula to appear paradoxical because he himself finds the problem of freedom insoluble, and thinks of that which determines the will as some kind of magic entity. His whole view, like that of many other defenders of freedom, seems to be invalidated by a desire to discover freedom in the wrong place, i. e., in choice itself. But choice is always determined, as we saw. Choosing, on the other hand, is free, and is the only thing free.

to be intrinsically worth while may weaken extremely powerful conative impulses which at the moment and in their own way are just as powerful as instinctive impulses. The artist may deny himself entirely in order to finish his picture. The martyr gives his body<sup>12</sup> to be burned. It is true that the instinctive impulses tend to be more constantly present with us, and that they may, for biological reasons, be most frequently of the greatest intensity, but it is not as a general statement true to say, with James and McDougall, that instinctive propensity is necessarily at any moment greater than Ideal propensity. If willing were a matter of mere impulse *versus* mere impulse, then it would seem a wiser course to say that sometimes the Ideal impulse would win and would therefore be stronger, and sometimes the instinctive impulse would. Human nature being what it is, the instinctive would probably win very much oftener. But to retain the notion of impulse *versus* impulse is, of course, to think incorrectly. Willing is not a matter of impulse *versus* impulse, but impulses become developed into sentiments, and sentiments developed into character. The opposition is therefore between organized *character* giving approval to an ideal, and a detached propensity. If the character is strongly and harmoniously organized, the tendency towards ideal choice may (because based upon a strong ideal sentiment) be permanently stronger than that towards more instinctive impulsions. As against the formulation of James, then, we should say that in the case of virtuous choice it was better expressed thus:

Ideal impulse as approved by character sufficiently organized by a strong sentiment for a single controlling ideal > instinctive impulse disowned by character.

<sup>12</sup> Of course, McDougall would say that these conative impulses are themselves instinctive. That is another point. Instinct is no doubt present, and important. But it remains to be proved that the category furnished is large enough to explain all moral choice.

I have now, perhaps, said sufficient to show what I mean by neutral freedom, which is the freedom of reflective choosing to be determined by one end or another. I now pass to the second type of freedom, (3) on our list. This is the freedom which is possessed by virtue, and is that to which St. Paul and Kant refer.

The question is as between the "higher" and the "lower" life. We are accustomed to say that the man who chooses to be determined by the higher ideal is freer than the man who chooses to be determined by the lower. Indeed, it is often thought that the latter in time practically loses even the neutral freedom he possesses, by sinking to the level of the beast who is unable to reflect. We have to enquire now what is the source of the meaning of this "higher" freedom?

The answer is, I think, that the experience of this higher freedom is simply the experience of an inner harmony which is dependent upon a man's having found his true vocation as a man. This higher freedom is a determination by law, but it is at once a cosmic law and a law of his inner nature, and through his oneness with the cosmos man experiences an uplifting, a power, a freedom which is difficult to describe, but which is certainly a fact. From the psychotherapeutic point of view, it is of the greatest importance that a man should, for full healthfulness of mind, discover an ideal which is satisfying, a function in which he can harmoniously express the whole of himself. It is the following of such an ideal, an ideal which satisfies, which produces this feeling of internal harmony which we call higher freedom.

In another, more negative, sense, this freedom is freedom from mere animal impulsion of the past. At the stage of "neutral" freedom, the stage of reflection, we emerge into a world of alternatives, unknown to the mere animal. By allowing ourselves to be determined by the more ideal

rather than the less ideal, we consolidate and develop this freedom to which evolution has brought us. We make into an actuality the possibility of something new and altogether higher which was before us at the stage of "neutral" freedom, and through the realization and creation of values we enter upon a sphere of reality utterly beyond the dreams of those who remain at the instinctive level. If, on the other hand, we do choose to be dominated by the lower rather than the higher, it is so much loss, and if we persist it may well be that we cease almost to be capable of reflecting, or of being sensitive to what reflection reveals. For this reflection, although all normal human beings possess the power of it to some degree, is, as we have said, a hard thing, and itself implies at least some sensitivity to the value of truth. The man, therefore, who says "be thou my good," to what reflection tells him is evil, in time becomes the slave of that evil on account of his growing tendency to be dominated by the leadings of immediate impulse rather than by the value of truth. That is to say, he may even lose in time the possibility of "neutral" freedom.

There is a somewhat different way in which the contrast between the slavery of vice and the freedom of virtue may be marked. It is seen by pointing out that the natural expression of the more intrinsic values that determine us is a creative expression, whilst the natural expression of instinctive impulse is limited, hereditary, and prescribed. We know that the conative tendency when confronted by danger is to run away, or possibly to become pugnacious. But no one can ever foretell just precisely how the artist is going to respond to the experience of beauty. So it is with goodness, for the goodness that is more than legalistic, conventional is in some sense spontaneous, creative. In the realm of creation there is scope, not only for different personal expressions of value, as between one subject and another, but the possibility of new and fresh individual

expressions in each case. The sensual man is a slave because he is chained in a cell within the prison-house of lust; his mind is dominated by one desire, and all his conduct is of a single kind. For the artist, whether in sensuous stuff or in the varied material which life presents, there is no beauty and no goodness and no truth which is ever small enough to be fathomed, and he may go on endlessly discovering new delights in reality, and may forever continue to clothe his joy in the many-colored stuffs of human expression. At the level of instinct where there is the everlasting sameness of repetition, where "the cuckoo says 'cuckoo' ten thousand times a day," the possibility of any alternative to sameness is inconceivable.

We must admit, then, that McDougall hits the mark when he now says that it is the fact of creation which proves freedom. Even the freedom of creation is not the license which some would seem to think an ideal; it is the determination by universal law. But this law of things is most prolific, for it demands not only that man shall know an objective value, but that he shall remake value ten thousand times and each time differently, by projecting it through the medium of his single individual unique personality.

I would conclude, not by summing up what I have already, for the sake of clearness, taken a good ideal of space to say in several different ways, but would for the moment return to the point at which I began. We said that reason tended to occupy a despised position in modern psychology. Whilst we must not urge of course that mere reason is enough to overcome temptation, we may re-emphasize two things. Firstly, the stage at which men reason is a stage at which they are able to contemplate objects more objectively than is possible at the stage of instinct. This power carries with it the power of being able to contemplate and to be influenced emotionally by the more

intrinsic values for their own sake. In this way the possession of reason is one of the factors which enable us to experience value, though more than reason is needed. Secondly, when we choose virtuously, our freedom of *non posse peccare* is a rational freedom in the sense that an internal harmony is attained by domination by a single rational principle.

LOUIS ARNAUD REID.

ABERYSTWITH, UNIVERSITY OF WALES.

## NEWTON'S OBJECTIONS TO DESCARTES'S ASTRONOMY

IMPORTANT as it is to know what ideas Newton borrowed, equally important it is to know what conceptions Newton reacted against and opposed, since by acting as an irritant to his speculation, these rejected conceptions served as a stimulant to further development of the conceptions he has taken over. The great negative influence of this kind on Newton was Descartes's physics and astronomy.<sup>1</sup>

Before the appearance of Newton's *Principia*, and even for some time after, Descartes's physics occupied the most prominent position among the learned men of Europe. The new Copernican system demanded a new celestial mechanics, which was apparently supplied by Descartes. This system was rapidly accepted, and in the English universities, it seemed to hold an undisputed sway with the exception of Cambridge, where it had been recently introduced by Isaac Barrow.<sup>2</sup> Newton was a pupil of Barrow. Newton studied and expressed his appreciation of the new Cartesian geometry,<sup>3</sup> which helped make Newton's immortal work possible.<sup>4</sup>

<sup>1</sup> For a positive ancestry of Newton's physics, see Snow, A. J., *Matter and Gravity in Newton's Physical Philosophy*. The Clarendon Press, Oxford. (In press.) Chapter I.

<sup>2</sup> Mabillean, Leop., *Histoire de la Philosophie Atomistique*, Paris, 1895, p. 153.

<sup>3</sup> Brewster, David, *Memoirs of the Life, Writings, and Discoveries of Sir Isaac Newton*, Edinburgh, 1855, Vol. 1, p. 22.

<sup>4</sup> Newton, however, never regarded Descartes' geometry as highly as he did Euclide's.

Why did Newton find it necessary to refute Descartes's system and construct one of his own? The reasons were many and diverse. First: Newton made a clear distinction between experimental philosophy and philosophy in general, and was always conscious of it in adopting any scientific supposition which by its very nature could not be *experimentally* proven. Descartes never made such distinction. Second: Newton thought that his scientific observations contradicted some fundamental principles of Cartesian physics. Third: according to Newton, Cartesian mechanistic philosophy not only failed to give a direct proof of God's existence, but really left no place for God.<sup>5</sup> Fourth: the corpuscular philosophy of Galileo, Newton thought, was, on the one hand, better adapted for a theistic doctrine of the cosmos, and on the other hand, its employment could, better than anything else, serve the purpose of describing the whole universe in terms of mathematical relations. We shall now consider the above-mentioned reasons.

Newton's first objection it will be more convenient to treat later.

In order to be able fully to understand Newton's second objection, it will be necessary to state Descartes' astronomical theory.

Descartes dealt only with two of the four scholastic categories,<sup>6</sup> quantity, (extension) and motion. Descartes gave up his early definition of motion—that is, "action by which a body passes from one state to another"—and adopted a new one, as he expressed it, "the transformation of one body to the body nearest it"—impact motion.<sup>7</sup> From this

<sup>5</sup> Of course, Descartes' proof of the existence of God did not lie in the field of physics.

<sup>6</sup> The scholastic categories are quantity, place, time, and motion.

<sup>7</sup> *Oeuvres de Descartes*, edited by Charles Adam and Paul Tannery, Paris, Vol. 8, p. 53; Vol. I, pp. 26-29.

definition of motion, he deduced his three laws of motion.<sup>8</sup> Seven rules follow these three laws.<sup>9</sup>

Descartes, as others in his day, regarded the sun as at the center of the solar system. The earth is not isolated, but, on the contrary, it is the center of a fluid matter, which is in a state of constant whirling motion—is a whirlpool or vortex. This fluid matter surrounds the earth completely, and it is this vortex which moves around the sun and carries the earth with it; the earth in reference to this surrounding fluid being perfectly stationary. Analogously, if a piece of straw moves with a stream of water, the straw is purely "passive" and stationary in reference to the water: the water, moving along, carries the piece of straw with it. A body is in motion when it will successively occupy the positions of other bodies nearest to itself, these in turn displacing others, until some sort of circuit is completed, each member of the cycle simultaneously moving into the place of the next. All this may be said of the motion of the earth. Then it follows from this description of motion that the earth is at rest if we regard the immediately adjacent bodies, for it is at the center of a fluid vortex moving round the earth, by which vortex the moon, or rather the moon's vortex (as explained below) is carried around the earth as center. But though the earth does not move with reference to the vicinity of bodies composing this earth-vortex—and thus it can be truly said that the earth is "at rest"<sup>10</sup>—the entire vortex, with the earth within it, does move, carried in a larger vortex round the sun, so that the earth changes its position with reference to the sun.

<sup>8</sup> Snow, A. J., *Matter and Gravity in Newton's Physical Philosophy*, Chap. I, Sec. 2. These rules were never understood, in all probability, not even by Descartes, who never regarded them as important to the understanding of Descartes' physics.

<sup>9</sup> Neither of these sets of laws is found in his work upon the world.

<sup>10</sup> *Oeuvres de Descartes*, edited by Charles Adam and Paul Tannery, Paris, Vol. 8, pp. 97, 98; 124, 125. For a diagram of Descartes' astronomy, see Vol. 8.

With this vortex hypothesis, Descartes attempts to explain the tendency of bodies to fall toward the earth if they are suspended in the air, providing that they are not hindered in doing so by other bodies. Just as any freely-moving light body—for example, a straw—caught in a whirlpool or eddy of water, will finally be whirled into its center, so any object freely suspended in the air of the “earth-vortex,” or small vortex immediately surrounding the earth and carried with the earth in the large vortex round the sun, will tend towards the center of the “earth-vortex.” The earth being the center of that vortex, quite naturally, all the bodies coarser or greater than the matter of which the vortex is composed, will fall to its center.

From the eclipses of the moon and of the sun, from the size of the moon, and from the intensity of its reflected light, we may infer that the moon is not far from the earth. Therefore, it is natural to suppose that the moon’s vortex is contained in the vortex of which the earth forms the center. And because the fluid matter of the vortex moves from west to east, it carries the moon along with it. But the circle described by the moon is larger than that of the earth, the earth making one complete revolution in reference to its axis in one day, while the moon takes twenty-eight.

When fixed stars become opaque, they cease to be permanent centers of any vortex, and they pass from one vortex to another which is immediately next to it, and so on still to another, continuing this movement; these stars we shall call comets.<sup>11</sup>

The solar system contains, all told, fourteen vortices. The vortex containing the sun is the largest of all; there are six vortices of the planets, one of the moon, four of the satellites of Jupiter, and two of Saturn. The planets

<sup>11</sup> *Oeuvres de Descartes*, edited by Charles Adam and Paul Tannery, Vol. 8, pp. 168-191; Vol. 9, pp. 172-190.

move more rapidly the nearer they approach the sun.<sup>12</sup>

As a matter of fact, this hypothesis is one of many possible ones, thought Descartes, insofar as the Creator of matter and motion display a gradation of relations. This hypothesis is not the first one nor the last one—and what the first one was like we do not know. Therefore, insofar as this supposition is not the primitive one, it is not the primary and true one, but rather a naturalistic hypothesis of physics.

Descartes supposed that fine corporeal substance formed extremely small bulbs, which are violently moving, and, among these, circulates fluid matter, which is still finer and moving still faster, and here and there, there are centers about which the vortices are formed and are constantly moving.<sup>13</sup> All celestial spaces are filled with this thin fluid matter, which forms into vortices with large masses of matter at their centers.<sup>14</sup> This assumption of space-filling, thin fluid was forced upon Descartes, he believes because of absolute necessity in astronomy. The evidence in astronomy makes such an assumption indispensable: action extending over enormous distances presupposes that space is "compact."

It was in relation to this astronomical theory that Newton raised his second main objection to Descartes—that his astronomy was contradicted by experimentally derived facts. He said that he endeavored to investigate the properties of the Cartesian vortices, in order to see whether the celestial phenomena could be explained thereby, and concluded that they could not be so explained, for the following reasons:

<sup>12</sup> *Oeuvres de Descartes*, edited by Charles Adam and Paul Tannery, Vol. 8, p. 197; Part II, p. 196.

<sup>13</sup> *Oeuvres de Descartes*, edited by Charles Adam and Paul Tannery, Vol. 9, Part II, pp. 324-5.

<sup>14</sup> *Oeuvres de Descartes*, edited by Charles Adam and Paul Tannery, Vol. 9, Part II, p. 325; Vol. 7, p. 89.

(A) Newton's objections involved the following astronomical data: the periodic times of the satellites revolving about the planet Jupiter are in the sesquiplicate ratio of their distances from the Jupiter's center; and the same rules hold true, with the greatest accuracy, between the revolving planets about the center of the sun. Therefore, if these satellites are carried in their respective vortices about Jupiter, and within Jupiter's vortex about the sun, then the vortices must also follow the same rule of revolution. But, as a matter of fact, that "periodic times of the parts of the vortex" are in the "duplicate ratio of distances from the center of motion; and this ratio cannot be diminished and reduced to the sesquiplicate," as the vortex theory would necessarily demand, unless either the matter of the vortex would proportionally become more and more fluid the further it is from the center, or the "resistance arising from the want of lubricity in the parts of the fluid, should, as the velocity with which the parts of the fluid are separated goes on increasing, be augmented with it in a greater ratio than that in which the velocity increases." But the unreasonableness of these two requirements is apparent. The "more gross and less" parts of the fluid will tend towards the circumference of the vortex, unless they be heavy; then, naturally, they would tend towards the center of it. And even when Newton supposes that the resistance is proportional to the velocity, though—as a matter of probability—the resistance is in a less ratio than that of the velocity, the periodic times of the vortex from the center will still be in a greater ratio than the duplicate of the distances. And yet some Cartesians—to make the case still more hopeless—think that the parts of the vortices move more swiftly at the center, then again slower as we travel outwards, and again swifter as we approach the "outskirts" of the vortex. And if it be so, then surely neither the sesquiplicate

(the ratio of the cube of a number to its square), nor any other uniform ratio can hold true of its motion.<sup>15</sup>

(B) If any small part of a vortex, whose particles are in particular relation among themselves, be supposed converted from a fluid to a solid condition, then this small part will move according to the same rule as before, since no change takes place in its density, "*vis infinita*, or figure." And again, if this solid part in a vortex, being of the same density as the rest of the vortex, be resolved into a fluid, this same part will again move according to the same law as it did in its solid state, but its parts, being now in a fluid state, will revolve and move among themselves. The motion of the "small part" will be the same as before, neglecting the motion of its individual parts, which do not affect the motion of the whole. This solid, now being like the other fluid of the vortex, will move as the rest of the vortex insofar as that is at equal distances from the center. Consequently, any solid, if it be of the same density as the rest of the vortex, will move with the same motion as the rest of the vortex does, "being relatively at rest in the matter that surrounds it." But if the solid part be more dense, established laws of mechanics enable us to deduce that it will endeavor to leave the center of the vortex, revolving out of the center in a spiral, and consequently would no longer be retained in the same orbit, and therefore, overcoming that force of the vortex by which—the Cartesians thought—the whole vortex is kept in a perfect equilibrium. It is also now apparent that a more rare part of the vortex, if it be solid or rare, must be of the same density as the fluid, in order to remain in the same orbit; but if it be so, it would have to "revolve according to the same law with those parts of the fluid that are at the same or equal distances from the center of the vortex." All this

<sup>15</sup> Newton, Sir Isaac, *Mathematical Principles of Natural Philosophy*, English translation by A. Motte, London, 1729, Vol. 2, pp. 195-6 [Book II, Section 9, pr. 52ff.].

is contrary to the Cartesian hypothesis, which assumes that solid bodies occupy the centers of vortices, and so we may conclude that the planets are *not* carried about the sun in *corporeal* vortices.

(C) The Copernican system, which Descartes claims he accepted, assumes that the planets describe ellipses in their revolutions about the sun, the sun being a common focus, and "by radii drawn to the sun, describe areas proportional to the times." But the motion of the parts of the vortex will not follow the above rule. Let three circles, of variant diameter, represent three orbits of revolving planets about the sun. The orbit having the greatest diameter is concentric with the sun, while the other two have independent centers, successively to the right of the sun, of variant distances from the sun, so that they will form an aphelion to the right of the sun and a perihelion to the left of the sun. If these three orbits are symbolized by the first three letters of the alphabet, each successive letter corresponding with each successive orbit in the order of greatness of the diameter, the circle with the smallest diameter will have its corresponding letter "A", etc. According to the rules of astronomy, the planet revolving in the orbit "B" will move more swiftly in its perihelion and more slowly in its aphelion, whereas, according to the rules of mechanics, "the matter of the vortex" ought to move faster in the narrow space, aphelion, and slower in the wider space, perihelion. Therefore, it is apparent that the rules of mechanics and astronomy contradict the Cartesian system of vortices. "So, at the beginning of the sign of Virgo, where the aphelion of Mars is at present, the distance between the orbits of Mars and Venus is to the distance between the orbits at the beginning of the sign of Pisces, as about three to two; and therefore, the matter of the vortex between those orbits ought to be swifter at the beginning of Pisces than at the beginning of Virgo, in the ratio of three to one. For the

narrower the space is through which the same quantity of matter passes in the same time of one revolution, the greater will be the velocity with which it passes through it. Therefore, the earth, being relatively at rest in this celestial matter, should be carried round by it, and revolve together with it about the sun; the velocity of the earth at the beginning of Pisces would be to its velocity at the beginning of Virgo in a sesquilateral ratio. Therefore, the sun's apparent diurnal motion at the beginning of Virgo, ought to be above seventy minutes; and, at the beginning of Pisces, less than forty-eight minutes. Whereas, on the contrary, that apparent motion of the sun is really greater at the beginning of Pisces than at the beginning of Virgo, as experience testifies—therefore, the earth is swifter at the beginning of Virgo than at the beginning of Pisces." Consequently, the vortex system of Descartes is "utterly irreconcilable with astronomical phenomena."<sup>16</sup>

(D) Because we notice the regular movements of our planets, Newton finds in this another objection to all-permeating—material—fluidity-of-space. "For thence it is manifest that the heavens are void of all sensible resistance and, by consequence, of all sensible matter."<sup>17</sup> Or, as he wrote on another occasion: "I cannot admit that a subtle matter fills the heavens, for the celestial motions are too regular to arise from vortices, and vortices would only disturb motion. But if any one should explain gravity and all its laws by the action of some subtle mediums, and should show that the motion of the planets and comets was not disturbed by this matter, I should by no means oppose it."<sup>18</sup>

<sup>16</sup> Newton, Sir Isaac, *Mathematical Principles of Natural Philosophy*, English translation by A. Motte, London, 1729, Vol. 2, pp. 196ff. [Book II, Sec. 9, prop. 53].

<sup>17</sup> Newton, Sir Isaac, *Optics*, London, 1806, Twenty-eighth query.

<sup>18</sup> Leibniz requested Newton's opinion concerning Huygen's material in the appendix, *Discours de la Cause de la Pesanteur*, to his *Traite de la Lumiere* of 1690. On Oct. 26, 1693, Newton favored Leibniz by giving the above-stated reply.

(E) It is evident from the motion of comets, that they could not be carried by a vortex, because they cut the orbits of the planets in all manners of angles and times.<sup>19</sup>

(F) Newton objected to making matter continuous, arguing that if it were so, free movement would be impossible. But, as a matter of fact, bodies do move and change positions in all sorts of ways; therefore, it follows that space is not full of matter, but there are also vacuums into which matter can move.

(G) If all space were equally full, "then the specific gravity of the fluid which fills the region of the air, on account of the extreme density of the matter," would be about the specific gravity of quicksilver, or gold, and therefore could descend in air. Bodies which are lighter do not descend in fluids.

(H) "And if the quantity of matter in a given space can [as Descartes says] by any rarefaction, be diminished, what should hinder a diminution to infinity?"<sup>20</sup>

It is owing to the development, by Newton, of the notion of universal gravitation, that Descartes' hypothesis of vortices falls, and with it the whole astronomical structure.

Newton's third objection to Descartes' mechanistic philosophy was theological. In this he agreed with most of the Cambridge men, who looked upon Descartes' mechanistic explanation of the movement of the planets, and of gravitation as pure and simple atheism. Though More felt theism to be safe only within the Cartesian mechanism, even he lodged strong objections to certain phases of Descartes'

<sup>19</sup> Clarke states the above as an objection of Newton, but we have to take this with caution. Clarke's notes to Rohault's system, Rohault, J., *Tractatus Physicus*, London, 1682, Part II, Ch. xxv, Note to Art. 22.

<sup>20</sup> Newton, Sir Isaac, *Mathematical Principles of Natural Philosophy*, English translation by A. Motte, London, 1729, Vol. 2, p. 224 [Book III, prop. 6, cor. 3]

mechanistic explanations,<sup>21</sup> while Boyle<sup>22</sup> and Clarke,<sup>23</sup> in defending and explaining Newton, protested strongly against the Cartesian "world machine."

Newton's objections as to this point were three-fold. In the first place, Newton objected to the possibilities which Descartes' system allowed of banishing God. It is true that Descartes, in order to have the atoms stable and imperishable, granted to God the function of continual creation. But Newton felt that Descartes's world-machine was so mechanically self-sufficient that the relation to it of God as continual creator was merely a superstructure. To Newton, God was so essential that His constant providential interference was necessary for all phenomena from the movements of the planets in their orbits to the history of a single atom. In the second place, Newton objected to Descartes' method of inferring metaphysical propositions from scientific propositions. Newton felt it essential to keep the spheres of metaphysics and of science distinct.

Newton did not want to rest the existence of God upon scientific deductions, feeling that that is not the way to get at metaphysical knowledge. Metaphysical<sup>24</sup> (philosophy in general) propositions are never deduced from scientific propositions, he reasoned, for metaphysical deductions follow the scientific ones only when *all* the scientific knowledge is known and ordered; they are the last deductions in a series of deductions. Scientific knowledge as we have it at present in its incomplete form is never metaphysical knowl-

<sup>21</sup> See More, Henry, *Euchiridion Metaphysicum sive De Rebus Incorporeis*, London, 1671, Ch. 11; More, Henry, *The Immortality of the Soul, so far as it is demonstrable from the knowledge of nature and the light of reason*, London, 1713. Preface, p. 12.

<sup>22</sup> Correspondence between Boyle and More in Boyle's collected works. *The works of the Honourable Robert Boyle*. A New Edition, London, 1772.

<sup>23</sup> *A Collection of papers which passed between the late learned Mr. Leibniz and Dr. Clarke, in the years 1715 and 1696, relating to the principles of natural philosophy and religion*. Knapton, London, 1717. See esp. pp. 151, 351.

<sup>24</sup> Newton never used the term metaphysics; our use of term here with reference to Newton is explained fully in Snow's, A. J., *Matter and Gravity in Newton's Physical Philosophy*. The Clarendon Press, Oxford (In press), Chapter V.

edge, from Newton's viewpoint. While science describes the secondary causes or the relations in mathematical form, it never is able to give us the real and final causes, which are metaphysical. Newton preferred dogmatically to infer the existence of God from certain physical facts, not as a scientific proposition, but rather in the form of a question, a suggestion which can not be scientifically proved—which can, at best, be asserted in the form of a corollary to physics.<sup>25</sup> In other words, Descartes' mechanistic explanation, although useful for isolated facts, is yet not a true *metaphysical* explanation for the cosmos as a whole, for a complete and sweeping knowledge of the universe would show it not as a mechanism but as a providential arrangement.

Newton never separated astronomy from the other sciences or mechanics from geometry: they were all one universal science, an experimental or natural philosophy, a unity of sciences. The real distinction that he made was between the sciences, or "natural philosophy," and philosophy in general, or natural theology. He applied the same method to all facts insofar as these facts could be controlled by one unity. Indeed, a science which is composed of disjunctive parts was for him not a real science, for it lacked unity and simplicity. This distinction, which allowed escape from theology, and experimental method were, of course, responsible for a great scientific progress. From then on, as we know, science began to develop at a previously unknown rate.

The world for Newton was geometrical but primarily theological. Newton thought that his *Principia* would point to the necessary existence of God.<sup>26</sup>

<sup>25</sup> Newton accepted God as the first cause, dogmatically holding that God being intelligent makes physical science possible because he guarantees the stability of the cosmos: it is the stability and the regularity of the universe that calls for the existence of a God who is its first and final cause. This contention receives justification and fuller treatment in Snow's, *Matter and Gravity in Newton's Physical Philosophy*, Chapter V.

<sup>26</sup> General scholium at the end of Newton's *Principia*; his letter to Bentley; Cote's preface to the second edition of the *Principia*. "Voltaire, toujours aussi

Newton's fourth and final reason for refusing to accept Descartes' astronomy was that the corpuscular or atomistic philosophy of Galileo, he thought, was better adapted to a theistic doctrine of the cosmos and because its employment would better serve the purpose of describing the whole universe in mathematical functions. Newton's reasoning was as follows:

(A) By adopting Galileo's and Gassendi's atomic hypothesis, Newton could very nicely employ it in favor of theism. If atoms are inert, then movement must come from other sources. Also, if matter is atomic and there are empty spaces, how can bits of matter attract each other across void? Surely not in terms of direct contact and impact motion, nor is it *intelligible to suppose they really attract each other at a distance*. It might be by some non-material means which moves and attracts matter. According to Descartes, movement is transmitted by impact, the only movement that takes place is that among solid bodies, and that is to say translations and rotations. The movement is conserved as long as it is transmitted in the form of shocks. But Newton thought that the movement due to the propagation of light cannot be explained in terms of shocks of solids, as will appear in chapter three.

(B) Descartes, in order to be able to think of matter in an objective sense, applied the doctrine of simplicity of nature, reasoning about as follows: If we try to think how things are constituted apart from the way in which they affect us, we come upon three characteristics and these are extension, divisibility, and mobility. These are the simplest and clearest ideas that we can have of matter, and if we think only of these, and these are purely geometrical qualities, we will be able to understand, by the use of logical deduction, all that happens in the material world. And if it

profond, s'écrivait que les athées n'avaient qu'à écouter le grand homme, qui démontrait Dieu en observant les astres." See Mabillean, Leop., *Histoire de la Philosophie Atomistique*, Paris, 1895, p. 446.

follows from the ultimate qualities of matter, that they are geometrical—that is, extension, divisibility, and mobility—then atoms in a strict sense do not exist, matter can be infinitely divided. Also if matter is primarily identified with extension, which really is space, then all space being matter is full, or, to put it in other words, vacuum cannot exist in nature. The material world must be infinite, for it is impossible to limit extension.<sup>27</sup>

Newton objected to Descartes' notion of matter, contending that only by identifying matter with purely geometrical characteristics is one led to believe that real atoms do not exist, and that matter can be divided indefinitely. Matter physically, Newton thought, has a limit to its division. Furthermore, as has been stated above, he believed that space is not full of matter, that there must be a vacuum into which bodies move when they change their position, especially when certain facts of nature enumerated above point to the existence of a vacuum. However, Descartes was close to atomism; for hard atoms, small round corpuscles may be substituted, which remain, as a matter of fact, unchanged, and are further divisible only potentially. In place of a vacuum Descartes has extremely fine splinters, which have been formed when the corpuscles were originally rounded.

(C) The hypothesis of individual finite atoms lends itself very nicely to a mathematical treatment of them. Galileo and Descartes defined matter in objective terms, while Newton gave to atoms a mathematical meaning, conceiving the simple atoms as points, and the relations of atoms to one another as geometrical relations. Newton could very easily employ mathematics in order to describe the material universe; the material universe is thus supposed to be made up of a large number of aggregates of atoms, the individual atoms being so small that their size would be

<sup>27</sup> More objected to matter being identified with extension. In fact, he thought that immaterial substance has also extension as one of its attributes.

negligible, and might serve very well as equivalent to the mathematical concept of a point. Starting in this way to conceive gravity in universal terms as an explanation of the movements of planets as well as of the fall of stones, the domain of mechanical physics is extended, and the world receives the formal enunciation of the principles of celestial mechanics, which have found their theoretical and practical utility to this very day. They are mathematical definitions of physics, models of nature, which can be modified as the necessity for it develops.

A. J. SNOW.

NORTHWESTERN UNIVERSITY.

## A NEW NATURAL THEOLOGY

### I

**I** HAVE recently published a theory of life which professes to perform the feat of "extruding the Subject." It claims to account for all the activities of animals, including man, without the aid of those "highly speculative" notions, Feeling, Sensation, Will, Emotion, Memory, Belief, Imagination, Knowledge, etc. ("Highly speculative" is quoted from a review of my book, "The Appearance of Mind," which is so described and damned.)

The theory has not yet attracted much attention, not unnaturally. It falls between several stools. It interests no established branch of science. Biologists take life for granted and study its processes. Psychologists take Consciousness for granted, and study the ways in which a Conscious Subject is related to its Objects. And philosophers do not notice scientific hypotheses until they are guaranteed by scientists.

I do not, however, propose to await the acceptance of my theory before proceeding to further study of its impacts, and one of the most interesting of these is on theology.

It is clear that a theory of life which did not profess to give an account of human religious feeling would convict itself at least of being only partial. Human religion is as factual as human noses. Incidentally, therefore, in the development of my theory I paid some attention to it. It

is a curious commentary on the present gulf between science and theology that my work is regarded by an American reviewer as "an attempt by the author to justify to himself his acceptance of both religion and the scientific method, rather than strictly a scientific work." Probably the occurrence of both "science" and "religion" within the boards of one book would induce a similar prejudice in the minds of most scientists.

## II

It is a common practice with biologists to regard living-activity as a series of reactions to stimuli. The response of the organism to environmental change is as primary for them as the Subject-Object relation for the psychologist.

My theory is an extraordinarily simple extension of this view of life. I regard the relation of an organism to its conditions as exactly analogous to that which exists between a chemical reaction, while it continues, and its conditions, a relation complicated only by the fact that the living-action never ceases, except in individual manifestations. A change in the conditions of a chemical reaction occasions a change in the reaction; for instance, it proceeds more briskly, as a rule, on the application of heat.

It is not my business to account for the fact of the occurrence of living-activity on this planet. I am given that life occurred and that it went on continuously to this day. I am even given, from the present point of view, which is especially concerned with "Mind," the fact of the evolution of "Body," for the vegetable kingdom exhibits an extraordinary variety of forms which have been evolved independently of "Mind." As a matter of fact, however, the evolution of "Body" does not need to be given; it follows from the mere assumption of an activity which contin-

ues forever (for practical purposes) through always-changing conditions. (Chemical activity, of course, has a "body," i. e., a material basis.)

A chemical action *naturally* changes with changes of its conditions; a change of conditions occasions a change in the activity. *Why* it does so is again not my concern, or at least not now. I am now concerned with the comparatively simple matter of exhibiting the real difference between plants and animals, the difference which is commonly supposed to be that the latter are "conscious," have each a "mind" of their own.

The real mystery of life is presented to us in plants no less than in animals, for it is not "consciousness." Neither is it adaptation to environment, which is simply a change in the very complex physiological activity of organisms in change of conditions. The real mystery arises with the occurrence of a special physiological habit of life, namely, that exhibited in the many-celled organism, the individual which sooner or later perishes, having been formed from the union of two individuals of the imperishable type.

This is a technical biological matter into the details of which this is not the place to enter. But the development of such individuals as ourselves from the egg to the mature type of the species is a phenomenon of which a theory of life must give some account, and in connection with which I may conveniently unfold my conception of the nature of living-activity.

The equilibrium existing between a living-activity and its conditions is characteristically different from that existing between a chemical reaction and its conditions in that the living action goes on forever. (In addition, of course, it must have been a peculiar and complex activity from the beginning.) In respect of *present* conditions there is not at first sight much difference between the two kinds of relation. A plant's adaptation to circumstances seems very

similar to a chemical reaction's adaptation, say, to increase of temperature. But it is more. For different plants react differently to the same environmental change. The equilibrium towards which they tend is different, is special to their type. It is a mystery, of course, but then so is that of the chemical action and its conditions, and in so far as the living equilibrium is more mysterious, it is easy to see whence the enhancement comes, namely, from the fact that living-activity goes on forever. Life adds the mystery of Time to that of Space. In the course of long ages variations of behavior have arisen and been perpetuated according as they were conducive to survival, or rather, as they were not conducive to non-survival. Thus the daisy opens at morning and closes at evening, and never dreams of staring at the sun all day like the sunflower. Each type of organism has its appropriate reactions to environmental changes, the "conventions" of the type, a word I use to emphasize the fact that a daisy's behavior is no more mechanical (a "tropism") than Professor Loeb's. It simply cannot be forced into the mould of natural law, i. e., of the laws of physics and chemistry. Tendency to the unique kind of equilibrium that obtains between living-activity and its conditions becomes, when the activity and its conditions becomes, when the activity has a long history behind it, "tendency to repetition in similar circumstances." The daisy does as its forbears did; what was good enough for them, it holds, is good enough for it. Life may exhibit a marvellous revolutionary variety of types, but within the type its manifestations are the extreme of conservatism. The living-reaction then of a planet to a *present* change is a very different kind of adaptation from that of the chemical reaction. It depends upon the racial history of the living individual.

Returning now to the morphological development of individuals, we may apply exactly the same considerations

to this special case, in which a tremendous amount of physiological activity goes on in the absence of any special environmental change to be its occasion. The important thing, we have seen, even in the reaction of an organism to *present* circumstances is its past. So it is with an egg. Its past determines it to develop into a chick or an oak-tree. Life in that particular type of manifestation tends to proceed in that particular direction. Tendency to equilibrium in these cases is tendency towards maturity of the type. "Action tends to be repeated in similar circumstances" is the sanction of the process of development, as of the behavior of the individual (or "as of the rest of its behavior."). If we wish to know what a hen's egg is likely to do, we have only to consider what happens in similar circumstances, what happened, in this instance, when the hen which laid the egg was an egg itself.

The name I give to the kind of equilibrium that obtains between living-activity and its conditions is "viable equilibrium." It is the tendency to viable equilibrium that is the special character of all living activity. Viable means "able to live," and when applied to conditions, "supporting life." Its use has hitherto been somewhat restricted, but I find that Stevenson had already rescued it from its purely obstetrical environment: "To judge by the eye, there is no race more viable; and yet death reaps them with both hands."

It is the tendency to viable equilibrium that determines me in my present activity. But there are many steps between physiological activity, which is all we have considered so far, and "highly speculative" (how that word rankles!) theorizing. Here I may be very brief on the evolution of behavior.

The first step is the evolution of motile forms of life, organisms which, merely incidentally to the processes of life, moved in some uniform direction, according to their

structure. This fact of locomotion, or locomotiveness, is not, in itself of great importance. But it is the pre-requisite of a very important novelty.

The epoch-making gesture was made when a motile organism, in presence of some unviable change, changed the direction of its movement. Caesar's crossing the Rubicon is by comparison a trivial incident in our history. It was the beginning of the Roman Empire. But this was the beginning of the Animal Kingdom. That organism *appeared* to be "conscious." And we do no more.

The appearance, I say, is that of a conscious subject, aware of a change, feeling its painful nature and willing to avoid it. The fact is the emergence of a new mode of achieving viable equilibrium. The animal organism, exposed to changes of environment, instead of adapting itself to them by more or less internal rearrangements, like the plant, changes its environment, not itself.

The animal, however, is not "susceptible" (and I may as well give notice now that I shall frequently use subjective terms without apology or even inverted commas—our language has been invented and used by organisms who thought they were conscious persons and to deny myself all these terms would make me tedious reading), the animal is not susceptible to all kinds of environmental change, only to those that most nearly touch him. It might have happened that we should have an electrical sense making wireless instruments superfluous. But electrical conditions did not touch our remote ancestors sufficiently nearly. They rubbed along all right with the rudiments of the five senses, more or less, and so we have just five. They avoided chemical irritants, they avoided extremes of heat and cold, they avoided mechanical disturbances and certain intensities of light. They avoided, in short, everything they noticed. They noticed only in that they avoided. It was their one

and only *psychological* reaction, if we may so designate this new mode of achieving viable equilibrium.

The primitive animal, like ourselves, was only "conscious" as it was "conscious" of change. And the only changes were enviable ones. Viability, for it, was the status quo. Its happiness was to be let alone, energizing, like a Stoic, "according to nature." Or rather, this would have been its happiness had it been conscious. But for this organism there was no conscious pleasure. All that it was conscious of was pain. It did not count its blessings. I assume that its nutrition was secured to it mechanically, as our oxygen is to us, incidentally to physiological functioning. It would only become aware of an unviable change if food-particles ceased to occur in its medium; and this would naturally be the occasion for it to go elsewhere. It would then present the appearance of seeking food.

In course of evolution the business of nutrition passed from the physiological to the psychological mode of functioning, i. e., the first part of the business, the procuring of food. The organism, instead of merely avoiding foodless parts and thus "seeking" those where there was food, now "sought" its particular food-objects. Its feeding-activity became positive, that is to say, the organism now had some "pleasure" in life. Incidentally, this was the beginning of that close involvement of the organism with the *things* of its environment which, as the relation of the conscious subject to its objects, is still the most controversial and the most central problem in philosophy. I simply decline to assume a "subject" "interested in," "attending to," objects. What I see is an organism which, given a certain occasion, exhibits an appropriate reaction.

Some types of animal organism have either very specific reactions or very specific objects or both. Others have a general reaction to a general type of object. This is the

distinction between the "small-brained" and the "large-brained" types of animal.

The development of the general tendency leads to "learning." For if an organism ingests a food-object which it finds "unviable," i. e., nasty, the appropriate reaction of course is to spew it out again. And this would occur every time, so that the general tendency would in itself be a disadvantage, an unviable variation. It is not, then, the general tendency which has preserved and developed the large-brained type of animal, it is the association with it of the capacity to learn. The law laid down that "Action tends to be repeated in similar circumstances" becomes qualified by another, "Unviable activity tends not to be repeated." This is the chicken's sanction for not pecking at red worsted every time it sees it.

The differentiation of the sexes is a technical biological matter upon which I shall not enter here, but ask the reader to accept my assurance that it presents no difficulty to my theory. It appears very far back in our ancestry, and when it has appeared the sexual appetite becomes as natural as the appetite for food.

The reader will see that the principle of tendency towards viable equilibrium is readily applicable to the behavior of the higher animals. The near neighborhood of a cat constitutes an unviable situation for a mouse,—also a terrier, another way. The approach of another dog disturbs the viable status quo of a dog engaged with a bone. A doubtful situation—perhaps painful, perhaps pleasurable—is the proper occasion of "curiosity." Parental care, with its sentiments, i. e., its organized modes of behavior, is an obviously viable variation, the development of which can be studied in all its stages, in vertebrate animals, from its almost non-existence in fishes which lay millions of eggs at a time up to its climax in mammals and birds.

The gregarious habit of life makes revolutionary changes in the animals which adopt it, whether small-brained or large-brained. In the former case it moulds their bodies, in the latter their "minds," their modes of behavior. Society made man, and makes him anew every generation, through his suggestibility. The viability of inter-subjective intercourse cannot be doubted, whether in a shoal of herring, a herd of deer, a flock of starlings, or a House of Commons.

The evolution of human hands (the paired fins of the fish), had a great deal to do with the evolution of man, as is recognized in the definition of him as "the tool-using animal." But most characteristic of man is the becoming articulate of inter-subjective intercourse, the giving of names to things, including the name "*Homo sapiens*" to himself. I should prefer "*Homo loquens*" or "*nominans*" as more accurate. ("*Insiapiens*" would be graceful humility in the wise, and correctly descriptive for the others.)

With speech emerges a new viability, that of Truth. Goodness had already emerged in the relation of the gregarious individual to his society. And Beauty—everyone likes things nice; neither speech nor social intercourse is necessary to the emergence of this spiritual value.

It is easy to see that the viability of such propositions as "There is corn in Egypt," depends upon their truth. A man who had travelled from Palestine on the strength of this one would be much injured if it were untrue.

It is difficult, however, at first sight, to see how propositions not of definitely viable import, i. e., academic propositions, ever came to be made. I have suggested that the first ones were negative, arising out of the observation of the unexpected. There is no reason at all why an infant, or man in his infancy, should remark "All swans are white." But let him meet a black swan. His status quo

is disturbed and his natural tendency is to cry "All swans are *not* white, then!"

The usefulness of even very highly academic knowledge, e. g., of the number of hairs on a beetle's leg, or of imaginary geometries, though now acknowledged by the educated, is not at all obvious to the vulgar. Such activities, however, like those of art, are not necessarily occasioned by the consideration of their possible ultimate usefulness. They are primarily the occupation of idle hands or minds. The unviability of nothing-to-do for organisms with active hands or "minds" is notorious.

I have now brought man up to date, and what is he? For practical purposes I have not the slightest objection to his calling himself a "conscious subject," a "person," with faculties ad lib. But I must emphasize that for theory he is nothing of the kind. The "person" is a "highly speculative" assumption. A man, like any other individual living thing, is a manifestation of life acting according to the laws of life. So much as a very brief outline of my theory of the evolution of the apparent conscious subject, an evolution, I may remark, that is implicit in the long-accepted theory of biological evolution.

### III

Probably the reader's first question will be "What becomes of Free Will?"

Strange as it may seem, a theory that seems stiff with determinism completely vindicates the doctrine of Free Will against the necessitarian. It denies the reality of the Will at all but upholds its freedom. In a word, it finds the uniquely characteristic feature of life in Contingency.

It is true that every living action takes place in accordance with law. But observe the nature of these laws.

"Action tends to be repeated in similar circumstances." Surely this leaves plenty of freedom of action. It states the freedom of the individual to choose his own fetters of habit. It is the very essence of self-determination.

To be sure, our instincts determine us to a large, a very large, extent. But will any man complain that he is determined by the tendencies that make him a man, and not a pig? The past cannot be undone. That certainly is determined. If a man is born a man, a man he must stay. The freedom of the will is maintained in that each *present* moment of the man's life is contingent, is not determined by his past. His action is determined only on retrospect, when it has happened, and not before.

In this, the individual manifestation of life resembles life as a whole. Evolution is a chapter of accidents. Whenever a biologist speaks of a new type of activity "emerging," he is referring to an accident—in such a case an epoch-making accident. The lion eats lamb and the lamb grass. They cannot do otherwise. But they had a common ancestor whose diet, whatever it was, could not have been both of these. The descendants of that ancestor, isolated by accident here and there throughout many varying environments and thus subjected to the necessity of adaptation, and themselves varying by mutation (also accident) have become different species with different natural tendencies.

By accident the ancestor of ourselves and of all animals turned away from unviable conditions, "not knowing whither he went." By accident, a later ancestor of ours forsook the straight path of the fixed reaction for the greater liberty of the general one. By accident another "became a sojourner in the land of promise," leaving the water. By accident, some creep, some fly, some walk erect. By accident some live alone, some in societies. By accident men speak. By accident they regard themselves as persons

and often as immortal souls. By accident—I hope, a happy accident—I deny the fact of personality. (I fear I fall short of the eloquence of my model, but perhaps accident is a less inspiring theme than faith.)

If the reader doubt whether instincts could arise by accident, let him consider the origination of habits. Instincts have been called "hereditary habits." It is just as permissible to call habits "acquired instincts." A man acquires habits according to his family, his society, his profession, his nationality, his race, according to all the varied aspects of his environment. But his relation to his conditions is the very essence of contingency. The laws of life can only determine his innate tendencies; they cannot determine the conditions in which his lot shall be cast.

Let it not be supposed that I profess to know positively what I mean by "accident." For my purpose it is sufficient for it to mean "not necessity." Rather than define "accident" positively, I should ask to be allowed to define life and say that it is that kind of activity which is not necessitated by present conditions.

Life is not in "Space" alone, physical activity is. A living thing is in "Space" and "Time" or better, is spatial and temporal. With all respect to the physicists, it is the prerogative of life, and of life only, to be four-dimensional. It is only for the sake of making the physical world what he calls "intelligible" that the physicist has given it four dimensions, or indeed three, without, however, as yet making it intelligible. "Time" and "Space" exist only for the human "mind." It is life that differentiates them out of formless chaos, and mankind that gives them names—but hardly meaning.

Only a living thing has a true history. The physical processes of the earth and of the stellar universe throughout millions of years no more constitute a true history than those of a chemical reaction in a test tube. They are only

larger and longer. Physical "events" are not events at all. They do not "happen." They just *are*. Happenings, events, occur only in the relation of a living thing to its conditions, because only living-activity is, to some extent, "free," this freedom from present conditions being the corollary to the fact that it is determined, to some extent, by the past. Life in its continuity affords the thread upon which history crystallizes.

These remarks on "accident" are designed to reconcile the reader, if possible, to that Cinderella of concepts, which is really disreputable only in comparison with the purely subjective concept of Purpose. The reader, however, is no longer allowed to have purposes. The choice left to him is—necessity or contingency? And however repugnant it may be to his self-esteem to suppose that he simply "happens," it must be more so to suppose that he is a mere machine, a highly complicated chemical equation. And if "Chance" reminds him chiefly of Monte Carlo and Epsom, let him remember also Pascal's wager.

A man then is free to be good or bad every moment of his life—not predestined a vessel of wrath or a vessel of mercy. He was created "sufficient to have stood, though free to fall." By accident, a happy accident, he stands; by accident he falls.

I may be supposed to have fallen here, myself—into confusion. I stated above that the freedom of the will was the essence of self-determination, and now I say that a man happens with luck to be good, or happens by bad luck to be bad, so that in neither case is he responsible. But it must be remembered that in theory I deny the Will altogether. And when I speak thus of a man happening to be good or bad, I am speaking strictly in terms of the theory.

My theory, however, is by no means alone in denying the possibility of real self-determination. All varieties of

Christian theology are quite clear that a man cannot save himself. "There, but for the grace of God, goes John Bradley."

#### IV

"For three things the earth is disquieted, and for four which it cannot bear"; for the man who thinks he is saved because he is within the law; for the man who thinks he is saved because he believes this and that; for the man who thinks he is saved because he likes things nice; and for the man who thinks he is saved because he is all these.

We saw that goodness and beauty are already viable, i. e., valuable, for animals below man—and we might have added Truth, for not only do animals detect falsehood, but intelligent and immoral dogs have been credibly reported to have acted lies. Now the persons referred to above are on this plane. It is true that a wolf has no idea of the saving grace that lies in believing that something happened, but he desires Goodness and Beauty with all his heart. He keeps within the law like a Pharisee, he worships Beauty with the ardor of a Ruskin.

The values of this world are vanity, even the highest of them. The man who identifies viable equilibrium with the attainment of beauty, of scientific or philosophic truth, or of the material welfare of his neighbor, the world, apart from the fact that he is not likely to attain these ends, would not be happy, if he did, as J. S. Mill saw in his moment of disillusionment. (How much less the man who thinks to attain it by means of purple and fine linen, sumptuous fare and a "good time.") To pretend that the happiness lies in the pursuit, not in the attainment, is a sham consolation. Are we to wander in the wilderness forever, merely making believe that there is a promised land? Or is it true that "the Kingdom of Heaven is within you"?

Surely it is. We became animals in the person of that motile organism which first "perceived" the unviability of its environment. To become "spiritual," to enter into the Kingdom of Heaven, is just as simple. It is to perceive "with awaken'd eyes" the unviability of this world at its best, to realize that viable equilibrium, happiness is not, for man, materially conditioned.

This is a simple fact which some people see and some don't. It can be discovered without depth in philosophy or searching the Scriptures. It is as well expressed "Money isn't everything" as in the most eloquent sermon ever preached on the text "Here have we no continuing city." And it has absolutely no connection with Truth, Beauty and Goodness, any more than our behavior has with pleasure and pain. Pleasure and pain are appearances arising out of behavior, out of the tendency to viable equilibrium. And so are truth, beauty, and goodness. They vary according to time and place; they are accidents of a man's conditions. There never were such impostors as these Absolute Values. As our knowledge is all illusion, so are these, the most abstract of our concepts, the climax of illusion. It is here that the philosopher and the theologian think they come nearest to reality, as the natural philosopher thinks he does with his atoms and electrons. Spirit and Matter are the last infirmity of learned minds. They owe their existence to the mind, and mind is an appearance.

A man may, by the grace of God, or by good luck, prefer the values that are not material. Such a man, the philosopher says, with the inveterate human tendency to rationalize, is a man who finds Goodness, Truth and Beauty in themselves desirable. But this is hypostatizing only. The objects and ideas of the mind are tainted with the unreality of the mind. Goodness, Truth and Beauty are idols; as absolute they are meaningless, as relative they serve no purpose. If we would make spiritual value real we must

make it lie in a relation, a relation between man and the universe. It emerges when a man's activity is no longer occasioned by material conditions alone (animal desires), when he attaches to these only their own proper value, and to others a higher value.

What are these others? A man who was asked what was his religion, replied "That of all wise men." "And which is that?" "No wise man says." It is no use telling those who know. And to others it is foolishness.

I have treated of large matters in small compass, perhaps unduly small. But my only purpose was to apply my theory to a special phase of humanity, a matter in which I could, and should, be brief, not to defend any of my own positions or to attack others.

In conclusion, I can be briefer still. The contingency found to be the special character of life ensures in man the freedom of the will. And reflection—and still more, perhaps, experience—discovers the vanity of material values or "viabilities."

J. C. McKERROW.

LONDON, ENGLAND.

## THE PROBLEM OF UNIVERSALS

THE need to distinguish between universals and particulars seems to arise thus. We encounter perhaps a triangle and a square both of which are red, and we recognize that they are identical in respect of a certain character which we call redness. Or perhaps we encounter two red triangles. Here are two objects which are identical as to attributes and different as to relations. We thus have to distinguish between this red, or this triangle, and the universal character of redness, or triangularity. This particular red triangle is limited as to position in space and time; redness and triangularity are not. Now this red triangle has unity. Redness and triangularity are intimately blended in this particular. But redness itself, and triangularity itself, has unity of another kind. Redness may be one and the same redness in innumerable instances. It does not exist, in the sense of being located in one space and time, and subject to causal relations. Redness remains redness whatever happens. Some would even persuade us that though there were never to be, nor to have been, red things, yet redness would be. Now in this red triangle, redness and triangularity are intimately blended; but not so intimately that the one cannot be without the other. For someone may blacken our triangle, and yet it will be a triangle. Or someone may clip its corners, and yet it will be red. But between some at least of the properties of triangularity there is a more intimate relation than between this redness and this triangularity. Thus, for instance, if we could increase the sum of its angles to 181 degrees, our

triangle would be no more. Thus it seems that of the connection of universals in a particular there are three possibilities. (a) Some characters cannot inhere in the same particular, such as squareness and triangularity. (b) Some necessarily occur together, such as triangularity and three-sidedness in figures. (c) Some may or may not occur together, such as triangularity and redness. But though a triangle need not be red, it must be colored, just as, though it need not be right-angled, its angles must sum to two right angles. Similarly, though it need not be made of paper, it must have some sensible embodiment, for even an imagined triangle is imagined as visible. And so it would seem that the various merely possible characters that a triangle may have are all alternatives limited by some wider necessity.

But having made this distinction between universal and particular, what shall we think of their relation? We may classify the possible approaches to the subject as follows:

There is in the first place nominalism, the doctrine that universals are nothing but names which we use to apply to many distinct individuals, "the doctrine that things called by the same name have only the name in common."<sup>1</sup>

Secondly, there is conceptualism. For those who hold that particulars alone exist in the external world may yet distrust pure nominalism, and argue rather that "concepts render possible a knowledge of real things when they are so formed as to correspond with the nature of the things."<sup>2</sup> This is the theory of *universalia post rem*, the theory according to which the universal is a mental entity derived from particulars and corresponding with all the instances from which it is derived.

Thirdly, we have the theory that universals alone exist, and that particulars are only highly complex universals, that this red triangle is redness and triangularity enter-

<sup>1</sup> Joseph. *An Introduction to Logic*, p. 31.

<sup>2</sup> *Op. cit.*, p. 25.

ing into relation with certain other complex universals which together form the spatio-temporal relations of this red triangle. This is the extreme form of the theory of *universalia ante rem*.

But fourthly, there is the modified form of the theory of *universalia ante rem* according to which particulars are instances which partake of, or copy, or are derived from, universal forms which do not exist in space and time, but subsist in a realm of pure being.

Fifthly, however, we have the Hegelian position, which is difficult to classify. For on the one hand, according to this view reality is pure form, pure idea, in the Platonic sense; and in this respect Hegelianism is a theory of *universalia ante rem*. But on the other hand all *our* ideas, all the forms that we apprehend, are said to be unreal abstractions from reality; and in this respect the theory is one of *universalia post rem*.

Finally there is the theory "which maintains the reality of 'universals' or characters the same in more individuals than one—of squareness as well as squares, justice as well as just men and actions, man-ness as well as men,"<sup>3</sup> but holds that these universals exist only *in* their instances, so that there would be no squareness unless there were squares, nor man-ness unless there were men. This is the theory of *universalia in re*.

Nominalism we need not discuss in detail, since, when stated strictly, it entirely fails to meet the fundamental objection that "general names are no mere means for abbreviating discourses, but their existence is grounded in what we must think the nature of objects of thought to be."<sup>4</sup> If things called by the same name have only the name in common, why should we trouble to distinguish between some things by giving them different names? If dogs have only the name "dog" in common, and this community arose only

<sup>3</sup> *Op. cit.*, p. 31.

<sup>4</sup> *Op. cit.*, p. 31.

from the need of economy in discourse, why do we not strike with the axe of economy, and decree that in future cats, winds and virtues shall also be called "dog"?

Passing from pure nominalism we encounter the theory that universals are mere abstractions from the rich particularity of existence. On this view we are supposed, as Professor Laird says, "to pare away the distinctive peculiarities of things, and so to be left with their common elements."<sup>8</sup> But we cannot thus generate universals from particulars. Paring off the coats of an onion only gives us less onion, "and in the end no onion at all." It is admitted that "it might be possible to arrive at redness by this process by eliminating the distinctive shades of red, but it would be interesting to know what color is when the redness of the reds and the greenness of the greens have been abstracted from it," or what triangularity is that is neither right-angled nor acute-angled nor obtuse-angled, or what the abstract man is that has none of the peculiarities of any man, woman or child, hermaphrodite.

But let us beware. Professor Laird has led us too far. If by abstraction is meant that the common properties of things are actually *taken out* of things, so that the abstract triangle and the abstract man actually exist, "in the mind" as self-complete and isolated entities, though abstractions, then we may laugh with him. Abstract man does not anywhere exist, even "in the mind." But on the other hand the characters which together are "man" in general, though they always *exist* along with other characters, may be *attended to* without regard to other characters. We have in fact been led too soon into a discussion of *universalia in re*. Let us defer this matter for a while. "Abstraction" may imply either *in re* or *post rem*, and it is with the latter view that we are now concerned.

<sup>8</sup> *Studies in Realism*, p. 110.

There is a kind of conceptualism according to which universality is "a function of certain particulars, i. e., of words and images and signs."<sup>6</sup> According to this "causal theory of thought" things which have similar mnemonic effects on us have the same names. This view is very much like nominalism; but it is not a whole-hearted nominalism, since it admits that things really are like and unlike. It admits further that we do really know something *about* things, even if we do not know things themselves, for we know the like and unlike effects that things have on us.

For Ogden and Richards, for instance, such words as "characters," "relation," "property," "stand for nothing beyond (indirectly) the individuals to which the alleged character would be applicable."<sup>7</sup> This is in accord with the Third Canon of Symbolism formulated by these authors, namely the Canon of Expansion,<sup>8</sup> which says that "the referent of a contracted symbol is the same as the referent of that symbol expanded." There is, they say, no need to assume a universal redness merely "because red things are every one of them red." The occurrence of similars does not compel us to recognize similarity, a universal. It is incredible that there should be "such universal denizens of a world of pure being." Semon, they think, is right in holding that abstractions are generated, after the manner of a composite photograph, from superimposed particulars.

But this will not do. The "apparent symbol," "relation," does not stand for all *individuals* that have relation. "Red" does not stand for all *individuals* that are red. "Red," for instance, does not stand for this red triangle nor that red rose. It stands for *the respect in which all red things are similar*. To deny the reality of universals is to deny that anything has any character. Abstractions will not arise from superimposed particulars unless those particulars

<sup>6</sup> *Op. cit.*, p. 11.

<sup>7</sup> *The Meaning of Meaning*, p. 154.

<sup>8</sup> *Ib.*, p. 193.

have certain points of identity. Association marries universals, but the universals must be there to be married. Association does not *beget* universals.

To hold that universals are the work of the mind is to admit that all that we know is the work of the mind. For all that we know is the characters of things.

Ogden and Richards do not really succeed in denying the reality of universals, though they try to do so; for immediately after condemning "redness" they admit that "all red things are red." And surely this means that red things are themselves similar in respect of a character, redness. To be true to themselves these authors should say that things are characterless; that they are nothing but bare particulars, occasions. But such a view is of course meaningless. Even the barest occasion must be an occasion of something.

It is true that we use words and other signs to save the trouble of producing their actual significates themselves. "On the other hand," says Professor Laird, "it is equally clear that these signs save trouble precisely because they signify universals and because these universals *therefore* apply to their particular instances."<sup>9</sup> To hold that words are identical with general facts is absurd, for the sound is the *symbol* of the quality, and not identical with it. In every statement we employ words according to the rules of a game, just as we do in mathematics. But also in both cases the rules are imposed by the nature of the external world, and the symbols express or mean the general facts that control them.

The third kind of theory which denies the reality of universals is that "according to which general facts are only the way in which existence organizes itself."<sup>10</sup> Professor Laird says that "this contention may take very different forms, but its primary significance is clearly in terms of

<sup>9</sup> *Studies in Realism*, p. 111.

<sup>10</sup> *Op. cit.*, p. 112.

the mind, and this primary significance is far more important than any other."

Now in Professor Laird's censure of the theory of universals as an organizing principle there seems to be some ambiguity. For if it be allowed, as it is by him, that we *discover* the organized content of our minds, and do not create it, then there seems no reason to assume, as he does, that organization is itself mental, and therefore that, on this theory, thinking produces the world. "The way in which existence organizes itself" is simply the way in which existence behaves or *is*. And thus we come once more to the theory of *universalia in re*, which is very different from the view that universals do not exist at all. In fact the "organization" theory of universals is bound to be either an "*ante rem*" or an "*in re*" theory, according as the organizing is conceived as the work of a cosmic mind or simply as the nature of existence. It cannot be a theory of universals as non-existent. But indeed no such theory can be fully and intelligibly stated, since it is an undeniable fact that things are like and unlike.

Let us now consider the view according to which the particular is a likeness or copy of a type which subsists apart from its instances.

It is argued that, though the particular existent's universal character is not derived from the mind, it is derived from something other than existents, from some logical entity in the realm of pure being. It is impossible here to do justice to this Platonic theory of forms. Any attempt to do so would lead us very far afield. I shall therefore only indicate in brief the objections to the theory, and at the same time I shall seek to show that what is true in it is in accord with another and simpler theory. Clearly, the reasons for this view seem to be two: (a) There is, in some sense, perfect squareness, but there are no perfect squares. Real squares approach that ideal, but never attain it. Where

then is that ideal? We may not (as we have seen) suppose it to be a mental creation, since if we do, we destroy the whole of knowledge; but if it is neither in the mind nor in existence, where is it, unless in a realm of pure objective idea? Further, since we do not derive squareness from squares, and since (it is argued) squares are only square by virtue of their approximation to the ideal squareness, we must conclude that what is fundamental is squareness, the universal. (b) On the other hand in every square there is something, squareness, which is other than, and more than, any given particular square, since it is both in this square and in that. Consequently, it is said, this something cannot by any means be merely derived from particulars. A number of particulars remains a mere particular, cannot be a universal. Therefore the truth must be that the universal is superior to, and prior to, the particular.

At first, certainly, it seems foolish to suppose that perfect squareness is given through sense. No one has ever seen a perfect square. All our figures are mere approximations to this ideal. Even when they seem to be perfectly square, we may be sure they have irregularities which we cannot detect. But on the other hand we need not suppose that perfect squareness has merely logical being. The square whose irregularities we cannot detect affords us an appearance of perfect squareness. It looks to us just as a strictly perfect square would look if we were to encounter one. And this appearance of perfect squareness is no mere mental construct; it is an experienced character. To deny its reality is as unjustifiable as to deny the reality of any illusory percept, to say for instance that the bent appearance of a straight stick in water is a purely mental entity. It is true that the physical objects that look to us perfectly square are not perfectly square. But, on the other hand, there *are* physical objects that are perfectly square. There are, in fact, infinitely numerous perfect squares upon all

surfaces. They merely wait to be traced by the perfect draughtsman. And were we to see one of them thus traced, and were we to be perfect measurers, they would afford us appearances identical with our common illusory perfect squares. But, further, were we never to see even an imperfect square, we might yet know the universal "perfect squareness." From our general experience of space we might infer the possibility of squares. This, however, does not prove that squareness can subsist apart from existent squares. It merely shows that we can infer the existence of figures which we have not perceived.

If this argument is true there is no need to derive particular squares from universal squareness. In fact, we should rather do the reverse and derive squareness from squares. But such a statement might mislead, for we do not *create* squareness as an abstraction (in the worst sense). We *observe* the square character of figures. Squareness, in fact, is in squares, and nowhere else.

We come now to the theory that universals alone exist, and that particulars are only highly complex universals. It is argued that this red triangle is nothing but a certain character-complex in certain relations. It is redness and triangularity having a certain size, and having certain spatial relations with other character-complexes. The redness that we experience in it is simply redness, which same quality we may experience elsewhere. Similarly with the triangularity. True, it is a special kind of triangularity, but still it is just a special quality which we may encounter a thousand times. Similarly the various relations of this red triangle are universals, identical whenever and wherever they are met. They are only particular in that we cannot within the great system of spatio-temporal relations, which is the physical world, encounter more than one instance of each of these relations. But an "instance" of a universal is, on this theory, nothing but the universal itself in rela-

tion with other universals. Two "identical" red triangles are not two particular existents in any other sense than that redness and triangularity share together in two distinct systems of spatio-temporal relations. There are not two such rednesses and triangularities, but one, which fulfills two sets of functions in somewhat the same way as one straight line may be at the same time a side of a square and a side of a triangle. On this theory the one-ness of the universe consists not in the existence of one great particular *instance* of a highly complex universal that might logically be repeated in other identical instances, but rather it consists in the fact that there is one all-embracing system of universals in which all universals are included, and all relations between universals. Duplication of such a system is, on this theory, meaningless; for the duplicate would either have relations with the original, in which case both would be universals included in a wider system, or it would have no such relations, in which case it would be absolutely identical with the original, and one and the same with it. Such a duplication of the universe would be as meaningless as the duplication of the universe of mathematics or the universe of the English language. Two systems of mathematics, or two languages, are only two insofar as they are different. If they are identical they are one and the same fact. This theory, in short, accepts the identity of indiscernibles.

It is commonly objected against this theory that it allows no distinction between percepts and concepts, and that this distinction is necessary. My percept of this dog, so it is said, is not the same as my concept even of *this* dog. Even if my concept were no mere fragmentary schema, but were complicated so far as to express every character of this dog, yet (so it is said) it would lack the infinitude and immediacy of the seen dog, and the dog, on the other hand, would lack the universality of the concept. To deny par-

ticulars, it is said, is as unintelligible as to deny universals.

But this objection will not bear close inspection. For on the theory before us the "particularity" of my percept "this dog" consists only in the relatedness of the universals of the dog with a certain spatio-temporal system which is itself universal. To be complete, the concept would need, not merely to give all the qualities of this dog, but all its relations; and then it would have the only possible particularity that even the percept can have. The objection, in fact, argues in a circle, basing itself upon the assumption of the very thing that it sets out to prove, namely the distinction between percept and concept. It is said that a second identical percept would be an instance of the same universal or concept, but would be itself a second and distinct particular. But the theory that is impugned merely retorts that if the two percepts really are identical then they are one and the same universal in two systems of temporal relations, or rather in different regions of the one all-inclusive system; whereas if they are not identical, then they are two universals. Whatever is identical in them is one universal; whatever is distinct is more than one universal. The theory before us holds that a percept is simply an observed instance of a concept; and that being an observed instance of a concept consists simply in the concept's being in relations (which are themselves universals) with other concepts.

But this suggests another and more serious objection to the theory that particulars are merely highly complex universals. For it is the very nature of universals, on this theory, to be supra-temporal eternal verities; how then do they come to *enter* into relations with one another, how do they *get into* the temporal order? If concepts dwell in a region of eternal truth, how is it that this dog is now running, now sleeping, or that I now observe him and now not? It is difficult to see how the same universal dog can be eter-

nally both running and lying asleep, both observed and not observed. It is all very well to say that the universal "running dog" is directly related with 10.30 a.m. on Christmas Day, 1923, and "sleeping dog" with 10.45 a.m.; but what of the "now" and the "not now"? It may be said that the universals that constitute me are related with those that constitute this running dog and with 10.30 on Christmas Day, but not with 2 p.m. But this seems to leave out of account the whole difference between present and not present, for 10.30 a.m. and 2 p.m. are not, on the ordinary view, entities that co-exist eternally; they are names for two states of the whole universe, two states which occur and vanish. We may, however, avoid this trouble by saying that time is no positive thing but an unreal appearance which is due to our own limitations, and that a better equipped intelligence would see 10.30 a.m. co-existing with 2 p.m., though not temporally, and the first of all days with the last of all days, though not temporally. But since in this matter we cannot go beyond guesswork, it is unwise to accept a theory which entails this view, provided we can discover another coherent theory which does not.

Here it is well to notice Mr. Russell's argument in support of the reality of particulars as distinct from universals.<sup>11</sup> He cites the case of two patches of white, identical as to predicates, but surrounded respectively by black and by red. The two patches, he says, are outside each other, and cannot be outside themselves. "It follows from this that the terms of spatial relations cannot be universals or collections of universals, but must be particulars capable of being exactly alike and yet numerically diverse."<sup>12</sup> This is the essential point of Mr. Russell's long and intricate argument, and to me at least it seems sufficient to refute any theory that would deny particulars. Yet, let us beware of following Mr. Russell in his cleavage of reality into "two

<sup>11</sup> *The Aristotelian Society's Proceedings*, 1911-12.

<sup>12</sup> *Ib.*, p. 17.

classes of entities" which respectively exist and subsist. Being, we must insist, is just being, and there are no kinds of it. If we set up these two classes of entities, we must state the relation of one to the other. And no one seems able to do this satisfactorily.

The American New Realists seem to derive existence from subsistence, or at least to hold that existence is a special case of subsistence. And Professor Laird, while distinguishing radically between particulars and universals, holds that "some general facts are logically independent of existence although existence itself cannot be independent of general facts."<sup>13</sup> We should, he says, be able to avoid the dualism of existence and subsistence if we could show "that all universals logically require particular instances which actually exist."<sup>14</sup> And he admits that many universals do require particular instances. "What kind of being could redness have if nothing were red, or what could sweetness be if there were no toothsome things?" But even in these cases, "though logic requires *some* existing instances," he points out that "it cannot deduce all the particular instances which happen to exist."<sup>15</sup> "The universal 'man' may logically require mankind, but this circumstance does not relieve the census officials."

But this may be questioned. *What we know* of the universal "man" is not sufficient to relieve the census officials; but we know only the barest outlines of that rich universal. Were we to know it fully we should have to say not merely that a man cannot be a true instance of the universal man unless there are more men than one (since man is social), but further that, this universal being what it is, the required mankind must be of such and such a various nature, and of just so many individuals living in such and such conditions. Similarly with color, we cannot agree with Professor Laird

<sup>13</sup> *A study in Realism*, p. 117.

<sup>14</sup> *Op. cit.*, p. 115.

<sup>15</sup> *Op. cit.*, p. 116.

that though "color" logically requires to have some varieties in the world, it does not logically require to have any determinate number of varieties." It is possible, but most fantastic, to hold that "color" and "man" and all conceivable and inconceivable universals subsist in absolute independence of any instances, that even if the physical and psychological universe were without color, yet color, the universal, would be. But once we grant the necessity of any instances we must admit the necessity of just those instances that are. So, at least, we must, unless we can remain in the logically uncomfortable half-way house of pluralism. And so, as with "man," we must say of "color" that if we knew this universal fully we should see that it involves all its instances.

But Professor Laird finds a more serious objection against any theory of *universalia in re*, and this objection makes him finally decide that universals are independent of existence. It seems perfectly clear, he says, "that two and two would be equal to four if no couples existed in the world," and further that pure mathematics "is logically independent of its application to existence, and so is the pure logic on which pure mathematics is based."<sup>16</sup> He has of course to admit "that *a priori* principles, like those of logic and number, do in fact apply to existence," but this, he contends, makes no difference to their priority to existence. The contention is in effect that though they apply to, or are instanced in, existence, yet they do not derive their being from existence. Professor Laird continues, "It is possible to maintain, it is true, that the general facts of number logically require *some* application to existence, although this application to existence is taken for granted throughout any demonstration in pure mathematics, and therefore does not enter into the demonstration itself."<sup>17</sup> But this view, which he has admitted in the case of "color"

<sup>16</sup> *Op. cit.*, p. 116.

<sup>17</sup> *Op. cit.*, p. 117.

and "man," he summarily rejects in the case of logic and number; for he says, "If this contention were sound it would meet the argument that two and two would plainly make four even if no couples and no quartettes existed; but it is impossible to see what grounds can be adduced in its favor; and it is clearly absurd to argue that pure logic and pure mathematics logically require existence just because they apply to existence."

But this is most distressing. We were given no reasons for *admitting* that color must have instances; and now we are given no reasons for *denying* that number must have instances. Yet this admission and this denial we are expected to make. If without red things we cannot have redness, how without numerous things shall we retain number? To the unsophisticated it would seem that number is nothing but a characteristic of numerous things, of members of an aggregate or parts of a whole. And Mr. Russell, if I understand him, accepts this view when he derives number from the similarity of all units and the similarity of all couples, and of all trios, and quartettes, and so on.<sup>18</sup> Merely to dismiss such a view as absurd is surely inadequate. It is not, however, to be contended that logic and pure mathematics require existence *because* they apply to existence; it is rather to be contended that they "apply" to existence because they are characteristics of existence, because without existence they are nothing at all, just as existence is nothing without them. What kind of being has two save as the common character of all couples, and four save as the common character of all quartettes? Were there no reds there would be no redness. True, and were there no existents there would be no number.

But Professor Laird's case may seem to be much stranger in respect of universals which never had or will have instances. Thus there are certain laws according to

<sup>18</sup> *Introduction to Mathematical Philosophy*, Chap. I.

which, let us say, Demosthenes *might* have been educated as an expert in distinguishing different pebbles by their taste. There can be little doubt that, in some sense or other, Demosthenes "might" have been so educated. But also there can be little doubt that not only he, but not any man at all, ever was or will be so educated. Of these universal laws there will, in all probability, be no instances. And yet in some sense or other the laws subsist. Their subsistence seems to consist in the fact that *if* the world had been in certain respects other than it is, and yet in certain respects the same as it is, then these laws *would* have had an instance. Conditional being is not the same as no being; and so it seems plausible to admit that after all these laws for the education of Demosthenes as a pebble-taster do subsist eternally, even without instances.

Now we cannot discuss this matter without raising the issue between monism and pluralism; and it is impossible to consider such a theme merely in passing. All that we can do is to note that whereas the pluralist must accept the conclusion that there are laws by which Demosthenes could have been educated as a pebble-taster, the monist, if he takes his monism seriously, cannot. The monist holds that the world is strictly systematic, that any change in a part involves change throughout the whole. He must therefore insist that, since there was in fact *no* Greek orator educated as a pebble-taster, the character of such an entity is incompatible with the nature of the universe, and in fact that in this world no such entity *could* have been produced. But if so, there are certainly *no laws* according to which he could have been produced. If as a superficial view there seem to be such laws, that is only because we cannot, in our ignorance, formulate any of the laws (supposed to apply here) sufficiently fully to see their material contradictions when applied to such a case.

For the purpose of this discussion, I propose to accept the view that the universe is fully systematic. And by this I mean that, given a change in some part of the universe, I can see no assignable limit to the changes correlated with it throughout the rest of the universe. I shall not here support this view beyond saying dogmatically that it seems quite clear to me that the whole trend of scientific discovery suggests that the more we know of the universe the more systematic it will appear, and that consequently, in the present state of our knowledge, the overwhelming probability is that the universe is systematic through and through. Pluralism and indeterminism, it seems to me, merely attribute to the constitution of the universe a lack of system which is due only to the limitations of our own knowledge.

Granting monism, then, we must admit that these supposed universals without instances, when fully known, would be seen to be inconsistent with the one and only coherent system of universals which is embodied in the cosmos. But, being inconsistent with the cosmos, they would finally be seen to be even in themselves self-contradictory. For they would entail characters of the cosmos, and yet deny the cosmos. That is to say that we must maintain that every so-called "universal" that can have no instances is an internally contradictory collection of universals, such as, for instance, "round square." For the trouble with "round square" is not merely that no-one has ever seen an instance of it, but that it is self-contradictory. And a self-contradictory entity is no one entity, not even one universal. It is two or more universals attended to in succession and never blended. Now the self-contradiction in "round square" is evident; but the self-contradiction in the great majority of pseudo-universals is not evident in the present state of our knowledge. Consequently while we are really apprehending two or more incompatible characters we think we are apprehending a single character

complex. It is generally supposed that there is a universal corresponding with the sign  $\sqrt{-1}$ ; but there is in fact no such universal. The  $\sqrt{-1}$  is self-contradictory. This however is an intermediate case, in which the contradiction is neither so obvious as in "round square," nor so obscure as in "the laws by which Demosthenes might have been educated as a pebble-taster."

It may be objected that, if the object of thought does not depend for its existence on being thought, we cannot think of nonentities, and therefore that we must conclude that these objects, such as round-squareness, subsist. But the true answer seems to be that we do not think of any such object as round-squareness. We think of roundness *and* of squareness. If it be suggested that after all there is a difference between thinking of, on the one hand, roundness *and* squareness, and on the other hand round-squareness, then our answer must be that it is just this difference that is important. For there is no contradiction in roundness followed by squareness, but there is contradiction in round-squareness. And while in fact we are thinking of roundness followed by squareness, we are blind to the "followed by," and so seem to be thinking of "round-squareness."

If the foregoing argument about Demosthenes is right we must conclude, as I have said, that there are no universals without instances. This, however, does not mean that "universal" is unnecessary, and that we can think of the world purely as particulars. For though there must be one instance of every universal, there may be more than one instance. This, it cannot be too much insisted, is the all important evidence for universals; or rather it is not merely evidence for universals, but the fact of universals.

But we are now face to face with a serious problem that has been haunting us for a long time and now refuses to

be ignored. We noticed long ago, but did not discuss, the view that all the forms that we know are unreal and deceptive abstractions. Reality on this view is pure idea, purely objective form; but all the forms that we are said to know lack self-completeness, for in reality they are all modified by an immense unknown factor. Reality does not consist of a patchwork of atomic forms known and atomic forms unknown. It is one and self-complete, and every element in it is penetrated by and fashioned by and sustained by the whole. No sooner do we abstract one form from the Absolute form, than it ceases to be what it was in the Absolute and becomes self-contradictory, unintelligible, and in fact a nonentity, or at best an entity whose whole being depends on its being thought by some limited subject. When, a little while back, we argued that every universal must have an instance, and that those supposed universals that have no instances are not really universals at all (since, if fully thought out, they would be seen self-contradictory), we laid ourselves open to the retort that all the universals of our thought are in principle self-contradictory when fully thought out, since they are ragged abstractions from the only coherent universal which is the Absolute Idea.

To this we must reply by distinguishing between full and adequate knowledge. For this objection only holds good if it is true that all our universals are inadequate, if, that is, the unknown factor which qualifies them in reality does in fact make them to be something different in reality from what they are in abstraction. If this is true, then clearly all our universals have no instances and we must conclude either that universals can be without instances, or that our universals are not universals at all. In either case our whole knowledge is vitiated; and we must cite the time-worn argument against all scepticism, namely that scepticism destroys itself.

But let us not be content with this facile answer. Let us rather question the truth of the view that all our universals are inadequate because in the real they are qualified by an unknown factor. It is hopeless to start with the presumption that thought is false. Both historically and logically the first presumption is that any given element of thought is true. It is only believed to be false when some other incontrovertible element or system of thought denies it. What then, we must ask, is the element or system of our thought which denies that our universals in abstraction are not identical with themselves as characters of the real? We are told in reply that reality is a true whole in which each member is what it is only in relation with and sustained by the whole, and that therefore when we abstract it it belies itself. But this answer needs to be qualified.

There are reasons for thinking that reality is a true whole, in that each part of it is what it is only in relation to the whole. But this does not necessarily mean that to be known it must be known in relation to the whole. It means only that to *be* it must *be* in relation to the whole. It is arguable that though the elements of reality are what they are only in relation to the whole, we may know something about some of them even when we do not know the whole. For they may appear to us as they really are, even while we cannot see how the whole makes them what they are. Thus we may know that straight sticks in water look bent, without knowing the laws of refraction. We may know the move of a knight in chess without knowing its relations to the moves of other pieces. We may know that  $2 + 2 = 4$  without understanding the *Principia Mathematica*, or even without knowing that  $2^2 = 4$ . In none of these cases do we know *all* about the subject, but in each case we really do know something; and there is no reason whatever to suppose that our knowledge is vitiated by the unknown factor, even though it is only in relation with the

unknown factor that the subject is what it is, and what it appears.

It may be objected that not only is reality a true whole, but even our knowledge itself is such that each element of it is what it is only in relation to the whole of our knowledge; and that, this being so, it is impossible for any element to bear the same relation to the whole of our knowledge as it does to the whole of reality; and consequently that we cannot know any element as it really is, unless we know the whole of reality. Now we have admitted that we cannot know *all* about anything unless we know everything. But we can really know something. Insofar as our knowledge is a true whole, it is *the* true whole of reality, but it is the merest schema of that whole. We may represent the whole of reality by a disc, and the knowledge of any individual by the circumference of that disc. Elements of the circumference are what they are in relation to the whole circumference, and also in relation to the interior of the disc. Thus the elements of our knowledge are what they are in relation to the rest of our knowledge, but also they are the same "what" in relation to the whole of reality.

It seems, then, that, even if reality is a true whole, there is no reason to suppose that all that we know is falsified by the great unknown factor. Some of our universals really are characters of reality, even though there is involved in them very much that we do not know. In such cases the unknown factor is not relevant to an adequate knowledge of the essentials of the universal, though it is relevant to a full knowledge of the universal in all its implications. On the other hand in any case in which the "universal" is found to be one that has no instances, it is one that is not only in conflict with the whole system of reality, but is also inconsistent with *itself* when fully understood. And such a "universal" is not a single universal at all, but several distinct universals to which we attend in sequence. Such,

as we have seen, is "round square," and  $\sqrt{-1}$ .

And so we conclude that the universals of our thought, when they really are single universals and not discrepant universals attended to together are really characters of reality; in fact, that every universal, that really is a universal, has an instance. We have thus no reason, so far, to reject the theory of *universalia in re*.

But there is an important matter still to consider; and in approaching it we shall accept the help of Mr. L. A. Reid, and, through him, of Professor Stout. Mr. Reid is among those who distinguish between universals and characters. He says,<sup>19</sup> "The character of this apple is to be red and hard and roughly spherical. These characters are not universals at all, they are particular existing facts and not a scrap 'logical.' When I know the red, hard, roughly spherical apple, *what* I know is a complex of particulars, or, if preferred, a whole in its particularized existing differences." But on the other hand, "when I say 'the apple is red,' the use of the term 'red' means that I am pronouncing this particular apple to be possessed of a particular quality which is a specific instance of the class or universal 'redness'." Thus "whatever and wherever secondary qualities are, they are particulars and *not* logical universals."

It is to comply with this distinction that Professor Stout has described the universal as a "distributive unity."<sup>20</sup> For he holds that the unity of a universal is a unity of a class, and that this unity is ultimate, and that any attempt to analyze it leads into a vicious circle. This view Mr. Reid describes and endorses. He agrees that characters are not universals, but "particular instances of the common class that has 'distributive unity'."<sup>21</sup> Says Professor Stout, "To say that particular things share in the common character

<sup>19</sup> *Knowledge and Truth*, p. 131.

<sup>20</sup> *The Nature of Universals and Propositions*, Proc. Brit. Acad., Vol. X, p. 4.

<sup>21</sup> Reid, p. 195.

is to say that each of them has a character which is a particular instance of this kind or class of characters. The particular instances are distributed amongst the particular things and so shared by them." Thus, while the universal has real being, the particularity of the characters, qualities and relations of things is also real. And, as Mr. Reid says, "in no case must the Universal be regarded *either* as simply beyond the particulars *or* as simply in them. It is not *beyond* the particulars because it has no existence apart from them." But on the other hand, "it is not *in* the particulars because what is in particulars is itself particular, a character such as a quality (or relation)." Consequently he concludes that "the terms '*in re*' or '*ante rem*' or '*post rem*' do not quite properly apply, for the universal is in none of these ways singly, related to the particular thing." But on the other hand he suggests that "if the universal is the concrete distributive unity of an existing class," it is inadvisable to say that it "subsists" rather than exists."<sup>22</sup> We might suggest that a fitting phrase for the description of this theory would be "*universalia in rebus*." This would escape the limitations of "*in re*," yet preserve the concrete reality of universals.

Now this concept of distributive unity was made necessary by the distinction between character and universal. And this distinction was made because there certainly seems to be a difference between "this red" and "redness," between any particular case of a character and the unity of that character in all cases. But to me at least it is not clear that this difference is an existential difference in the objects; it seems to consist rather in a difference of relations of one and the same entity. When we consider the "red" of this rose, we consider the same entity as we consider when we consider "redness"; but we are concerned with its present spatio-temporal relations. When, on the

<sup>22</sup> Reid, pp. 198-9.

other hand, we consider "redness," we are concerned simply with "red," but with the identity of many particulars in respect of red character. And this only means that we consider as one object many indistinguishables. In both cases our object is a "character," but in the one case we observe certain spatio-temporal relations in which that character occurs, while in the other we observe its identity relations.

But "distributive unity" is certainly a useful concept. It emphasizes the fact that *identity* of character involves more than one instance, while on the other hand the universal is not something above and beyond its instances. The unity which, it is said, is the universal, certainly seems not to be in any particular alone. Nor on the other hand is it a link between particulars; nor is it a pattern to which they conform. It is just the unity of many particulars, the distributive unity of a class.

But we must not forget that though the unity is not, as unity, in any one particular apart from others, it is in some sense in them all. If the universal is mere unity, and not unity in respect of some quality, it is not in each particular. But it is not mere unity; it is unity in respect of a certain character, and the character must occur in every instance. When we think of the universal we think not of mere unity but of the character, which admittedly inheres in particulars. For the unity of all "reds" is not the same unity as the unity of all "bads," or "sads" or "sweets." We cannot think of the unity of "redness" save as of "something red," though the something is indeterminate, a host of diverse particulars whose characters cancel one another save in respect of redness. In fact, we may say that the universal is a *character* of a class or whole, namely that character in which all the members of the class are identical.

We must then conclude that in spite of the useful concept of "distributive unity," the universal is in an important

sense *in* each of its instances. For the universal is not merely unity, but unity in respect of one character. This is very important; for it means that though the particulars do not share in a type, or form, which is distinct from them, they do *share in one another*. Each is identical with others in a certain respect. This fact seems to point to monism. Were there no identity at all in particulars we might believe in their absolute separation as self-complete individuals. But since there is identity, since they do share in one another, we seem forced to hold that the being of each is involved in the being of others, and indirectly with the being of all.

In philosophy extremes are apt to meet. And here we find the theory of *universalia in re* becoming curiously like the theory according to which particulars are nothing but highly complex universals. By insisting that the universal is in the particular we seem to have left no room in the particular for its particularity. This leads us to much the same position as that of the theory which abolishes particulars and tries to make universals do their work. But there is an important difference between the two views. For the theory that would abolish particulars seeks to abolish particularity altogether. The universal that it offers us is, in the worst sense of the word, an abstraction, a "what" divorced from every "that." But our universal is more concrete. It is a character which, to have being at all, must occur in one or more instances. It cannot be divorced without losing its being. And so, since reality is nothing but instances of characters, it is nothing but universals—in our sense of the word.

O. STAPLEDON.

LIVERPOOL, ENGLAND.

## THE CRUCIAL PROBLEM IN MONADOLGY

THE theory of Monads rests on the frank recognition of two facts. The first is that the knowledge which any individual subject of experience possesses is solipsistic. The second is that the relation between one mind and another is an ideal relation and there is no real contact. The first fact was expressed by Leibniz in the dictum: "The Monads have no windows by means of which anything can enter in or pass out." (*Monadology*, 7.) The second was expressed by saying that the Monads exercise an ideal influence on one another. (*Monadology*, 51.) It follows therefore that Monads cannot interact and also that they can intercommunicate. The crucial problem in Monadology therefore is: How can there be intercourse or intercommunication of ideas between Monads if knowledge is solipsistic and there is no interaction?

### I. IS THERE AN INITIAL SELF-CONTRADICTION IN THE THEORY OF MONADS?

The theory of the monads, despite the attractiveness it possesses, is usually dismissed summarily on the ground that its two basal propositions are a self-contradiction, and moreover that this self-contradiction is itself two-fold. For first, if knowledge is solipsistic it seems impossible that a monad can know there are other monads. Secondly, if interaction between monads is denied, it appears a plain contradiction in terms to affirm that one monad can exer-

cise an influence such as intercourse pre-supposes. The logical dilemma in each argument appears complete and yet as a matter of fact both arguments are and have always proved to be strengthless.

## 2. THE SENSE IN WHICH THE UNKNOWABILITY OF A THING MAY BE AFFIRMED

The first argument is practically identical with the objection which is continually opposed to Kant's doctrine of the existence of an unknowable thing-in-itself. We cannot, it is said, affirm that a thing exists and also that it is unknowable, for if it is unknowable not even its existence is known; nay further, it is only possible to affirm unknowability of anything if we are already in possession of some knowledge about it, and therefore to affirm unknowability is to deny what we are affirming. Obvious as this argument is generally held to be, it is not effective. We feel that the contradiction can only be verbal for Kant's doctrine is intelligible and may be true. The answer to the objection is that self-knowledge, even in the extreme solipsistic meaning, contains within itself the implicit distinction between knowing a thing as it exists for the knower and knowing a thing as it exists for itself. Even in ordinary common sense experience we make this distinction. We recognize that to know there are other minds is not to know other minds as they are in themselves, and we have no difficulty in reconciling our knowledge of the one with our conviction of the complete unknowability of the other. The argument is an empty logomachy.

3. IS EVERY RELATION A FORM OF INTERACTION OR  
DEPENDENT UPON AN INTERACTION?

The second objection is equally ineffective. It seems *prima facie* absurd to affirm that monads which by definition are reals, that is, things-in-themselves and not mind-dependent ideas, do not interact and yet nevertheless can enter into the relations with one another which intercourse implies. The reason simply is that we have come to regard all relatedness as being essentially a form of interaction. Interaction, however, has a quite unambiguous meaning. In physics it is the fundamental conception. It is defined by Newton's third law of motion. It is the concept which underlies the principle of conservation wherever there is physical action and reaction. It affirms a perfect equivalence or quantitative quality between every action and its consequent reaction. The principle was originally applied by Descartes to movement, which he held to be constant in quantity and indestructible. Leibniz demonstrated the defect of this application and proved that it could only apply to what we conceive as force or energy, and the conservation of energy has since remained a fundamental principle of physics. In every transformation of energy from a latent form to a kinetic form or from a kinetic form to a latent form we postulate an exact equivalence. If we keep to this meaning of the term interaction it is obvious that the principle does not apply to psychical relations even by analogy. In the daily intercourse which is part of the routine activity of individual human beings there is no interaction so far as the interchange of ideas is concerned. Minds are not impoverished when they impart ideas, on the contrary they are, though in a special meaning of the

term, enriched; and minds which receive ideas do not in the literal sense receive anything, for a mind can only have its own ideas and these must originate within it. There is then in ordinary experience a relation between individuals which is not interaction.

Though this is true, however, it by no means solves our problem or lightens our difficulty. If the exchange of ideas is not itself an interaction it seems to be conditioned by what is an interaction and to be wholly and necessarily bound to this condition. In order to have an interchange of ideas there must be speech, either articulated or written, and speech is an interaction, not indeed between mind and mind, but between the organized sense-receptive bodies of the individuals, which bodies are continuous with and part of the physical world. If I would impart an idea to you it is true I do not pass something out of my mind leaving it poorer, and you do not receive into your mind an idea which is not your own idea, yet I can only effect this intercommunication by initiating a purely physical interaction between our bodies through the medium of the external universe to which the principle of conservation applies down to the minutest detail. This is the condition of intercourse and it seems to carry as its natural corollary that if we accept intercourse as fact we thereby postulate a world which is common to both of us and independent of each of us. Consequently, common-sense assumes, and seems abundantly justified in assuming, an external world, and physical science seems to find it necessary and essential to postulate a self-existing material universe in space and time, indifferent to the truth or error of our representations of it and of our thoughts about it. Were there no alternative the theory of monads would stand condemned by the force of plain fact. There is an alternative and the theory of monads is itself the alternative. The essential point in the theory is its denial of the assumption and its deliber-

ate rejection of the scientific postulate. The theory of monads is that the conception of a common universe is a consequence of monadic intercourse and not its condition. It is from intercourse that the concept arises, out of intercourse that it takes shape, and not *vice versa*. The monads are not, as they are so often misrepresented, merely private perspectives. It is the monad which is private, not its outlook. The monad has no outlook. The monads are the reals and there is no universe except the universe which the monads themselves constitute. It is essential that the full significance of this aspect of monadology should be comprehended. It means that there is an alternative to the common-sense and scientific view that there must be a common universe, self-existent, as the pre-condition of mental activity. It is that the external world is an illusion which necessarily arises as a direct consequence of minds communicating. In the monadic theory, for example, space is for each monad the order of co-existences in its active perception, time the order of successions. There is no need to assume, nor reason to believe in, a space in itself or a time in itself or matter in itself, but if and when intercourse takes place between the monads the idea will naturally arise that the different orders of each monad to which each in its intercourse refers are one order common to all. If the metaphysic of monadism is consistent and well-founded, then the postulate of an external world is otiose.

#### 4. MONADOLOGY AN ALTERNATIVE TO MECHANISM

Monadology then is a metaphysic of reality. In challenging the common-sense view it claims to be itself consistent and self-contained in its integrality. It does not pretend to harmonize contradictions, what it does is to point to the contradictions of the common-sense view as

evidence that common-sense is concerned with partial aspects of reality. The principle of individuality requires us to conceive the whole in its unity and this unity as indivisible. We may take an organism and disintegrate it, dissecting out what we may choose to consider its constituent parts, and represent these constituents as independent elements of which the whole is the aggregate, but in dividing the unity we lose the reality.

The alternative principle of monadology in regard to the genesis of the idea of an external universe is no longer a principle of philosophical idealism opposed to a principle of positive science. It is identical with the principle of relativity which we owe to the genius of Einstein. According to the principle of relativity the external world to which we refer in physics is not a discovery and is not a postulate and no scientific advantage is to be gained by assuming its independent existence. We may even reject an absolute system of reference as both self-contradictory and as disproved by positive experiment. The principle of relativity is that the laws of nature are uniform for observers in different systems of reference moving relatively to one another, on condition that there are equations which prove invariant when transformed for the relative movements of the systems. The mathematical demonstration of these invariant equations is the scientific triumph of the modern world.

##### 5. LEIBNIZ'S REJECTION OF LOCKE'S THEORY OF THE ORIGIN OF IDEAS

I will now give what seems to me the direct answer to the question: How can there be intercourse if knowledge is solipsistic? Leibniz's own view is stated with particular clearness and with full emphasis in the beginning of his *Nouveaux Essais*. (Bk. I, Chap. I.) Philalithe, repre-

senting Locke in the dialogue, begins with a brief description of the general nature of Locke's argument in opposition to the doctrine of innate ideas and of his claim to have demonstrated the uselessness of the doctrine even as a hypothesis since man is able to acquire all the knowledge he possesses "without the aid of any innate impression." Theophite, representing Leibniz himself, replies: "You know, Philaethe, I have long been of the contrary opinion. I have always held with Descartes, and I hold still, that the idea of God is innate, and that there are also other ideas which must be innate since they cannot be derived from the senses. Now, however, in accordance with my new system I go still further; and indeed I believe that all the thoughts and actions of our soul arise from an internal source within itself, without it being possible for its ideas to be given to it by the senses."

This last sentence expresses the fundamental concept of Leibniz's philosophy and it is essential to give it full weight in constructing the scheme of monadic intercourse. We have only to understand the nature of ideas in order to see at once the absurdity of imagining that they originate outside the mind and are mechanically conveyed into it by the avenue of the senses. (I use the term *idea* to include every form of mental existent—images and concepts, representations and thoughts.) No doubt, Leibniz goes on to say in the same context, the senses do in a certain manner originate ideas by being the occasion of our perceiving, but ideas are not, like the "intelligible species" of the scholastics, cast off by external objects, nor are they a kind of impression on wax, or marks on a clean slate. They are forms which the mind itself, by reason of its active nature, imposes on its experience. They are innate in the sense that they depend on the monad's activity, on the degree of clearness and distinctness in its perceptions, and on the final causes of its actions.

Such is Leibniz's view of the nature of ideas in conformity with his theory of monads. It is generally thought that when he added to the famous dictum *Nihil est in intellectu quod non prius in sensu* the qualifying words, *nisi intellectus ipse* he was pointing out an exception. In fact, he was completely altering its whole import. Modern biology and psychology have come to confirm Leibniz's doctrine and have given it an illustration and a meaning which it was impossible for him to foresee. He held indeed that every monad is qualitatively distinguished from every other, but for him the monadic activity consisted in perception alone, and as regards perception he could only conceive a difference in degree. Intellect appeared to him the exercise of a faculty of reason which in itself was absolute, a view implicit in all philosophical theories previous to the concept of organic evolution. It could never have occurred to Leibniz that intellect might be a product of biological evolution. For him intellect characterized the higher monads, the monads who are souls, and it took the form of apperception as distinct from pure perception, or as we may render it, self-consciousness as distinct from consciousness. It was by apperception that the rational monad had the power to form the idea of other monads, for while mere manifoldness is given in the unity of pure perception, in apperception we experience and know the in-itselfness of existence. (*Monadology*, 16.)

## 6. THE IN-ITSELF NATURE OF THE MONAD

The monad then is the concept of the individual, philosophically defined in logically precise terms as an indivisible unit center of activity, self-contained in its actions, expressing in its actions its own nature. To take an explicit example let us consider the individuality of a

human being. A human being enters on its separate existence in the form which, as presented to our external perception, we term a fertilized ovum. Internally and in-itself it is a heritage of instinctive tendencies and developing potencies. It will unfold and expend its inner force in response to and subject to favorable conditions of environment, yet its whole energy and its directing power are innate and self-contained. How then in fact does such a solipsistic individual come to recognize other individuals and inter-communicate with them? The usual answer is that it must arrive at this recognition by reasoning on its sense-experience. This is transparent nonsense. Recognition is the outcome of the individual's own power of self-expression and of the sympathetic or antipathetic response it meets. This is the clue to the monadic scheme of intercourse which can now be set forth in abstract terms. The monad being conceived as activity must express itself in actions. Actions are innately determined but externally limited. The external limitations can only be perceived from within as self-limitations, but if the limitations are beyond control they will be conceived as alien forces. The limitations to the activity of a monad and the range they allow to its expression in actions give form and outline to the world. The world for the monad is the range of its activity. If now we conceive this activity as consisting in perception alone, the objects perceived will be the limitations or determinations of its actions reflected back to the perceiving center. In the simple monad there is nothing more. Its activity is pure perception and it is self-contained and self-sufficient. But besides the simple monads there are the higher monads with greater powers. Some monads are entelechies or animal souls and some are rational souls. To the rational soul or apperceiving monad something more is possible, for apperception is self-consciousness and therefore knowledge of in-itselfness, and therefore the rational

soul, without transcending itself, without disruption of its essential unity, can recognize in-itselfness as pertaining to the activities opposed to it. Thus it is self-consciousness which opens to the monad the possibility of knowing the existence of other monads. Leibniz indeed introduced at this point his famous theory of pre-established harmony. To me it seems, in the light which modern science throws on the problem, entirely unnecessary. The monads are not juxtaposed in space, and therefore harmony and not interaction expresses the nature of their relations, but the *deus ex machina* of a pre-established harmony is uncalled for. The harmony is not pre-established but follows from their nature.

#### 7. THE IMPLICATIONS OF APPERCEPTION OR SELF-CONSCIOUSNESS

To sum up the argument, both Leibniz's and my own, it is that the apperception or self-consciousness of a monad opens to it the possibility of knowing that other monads exist. Let us suppose the possibility realized and the knowledge attained. The crucial problem still remains. How is it possible for a monad to have intersubjective intercourse with another? The best approach to the problem is to consider how human individuals do actually hold intercourse. We communicate by speech. Speech is an acquirement conditioned by the development in the human organism of a complex neuro-muscular system of articulation, correlated with a sensory organ receptive of sound. Before such a contrivance can have been constituted as a means of intercourse the individual must have been able himself to express to himself the meaning which speech is to convey to another. Speech or any device for intercourse presupposes therefore that the individuals who would com-

municate by means of it have already found expression for their intuitions. This is the cardinal point. Human speech is the outcome of man's artistic nature, of his power to use material to create forms into which to fix the expression of his intuitions. This artistic creation is the source and condition of intercourse. When a man makes use of a plastic material in order to express his artistic intuition, as when he fashions a stone into what for him is the semblance of a man, there is nothing absolute in the material object or in the form impressed on it, the stone is simply employed *ad hoc* for an expression and a conveyance of meaning. The meaning it expresses to the artist who has fashioned it is conveyed to the man who beholds it only to the extent that the beholder has the creative artistic power to make it express his own intuition. It conveys no meaning to a bird or to a dog, nor has it even the power to deceive them by a semblance. The bird may find it a convenient resting-place insensible of the human form into which the artist has fashioned it. Another human being may interpret it, not on account of anything absolute in the form or in the material but because his mentality is attuned to the artist's mentality and solely to the extent to which it is attuned. Human beings, in short, are able to be responsive to the expressions of one another because they are organized on the same type, confined to the same range of action and identical in their life interests.

The reception of a communication, then, supposes in the receiver a process of artistic creation which originates within him, and the conditions of intercourse are, first that the intercommunicating minds are by their inner nature already attuned to one another, and second that each communicating mind acts creatively for itself. In ordinary human intercourse there is no interaction. If a man has an experience to which he cannot find means of giving outward expression, he cannot communicate it; and if he can

express his intuition in image or concept and convey it by speech it is only on condition that his self-expression evokes responsive self-expression in the mind which receives it.

#### 8. WHY THE POSTULATE OF AN EXTERNAL WORLD IS USELESS

It will be said however that I have labored to evolve a round-about and difficult scheme through prejudice, and in an effort to avoid the one simple condition which makes the relation of intercourse possible and perfectly easy to understand. We have only, it will be said, to assume the compresence of two minds in a universe the laws of which condition both, and this of itself will afford them a common ground of reference. Where but from such a universe, I shall be asked, does an artist obtain his plastic material in which he fixes the expression of his intuition, be his material a pigment, or a tone, or a solid resisting stuff? My reply is that existence of such a universe and the compresence of monads within it, even if it be fact, is irrelevant. It would present us with a new problem but in no way assist us to solve our present one. Let us suppose that there is an external world common to the monads and existing independently of them and of their activity of perception, how will it explain intercourse unless we already know the means by which it would or could reveal itself to the monads and itself produce to them the evidence of its in-itself nature beneath the varied appearances it must present to them? The only question about the hypothesis of an external world which is relevant is whether, seeing that it is impossible we can know by direct evidence that such a world exists, it is necessary in the interests of practical life or advantageous to physical science that we should assume it. To this question monadology is itself the clear and

unambiguous answer that the assumption is unnecessary and the postulate useless. If intercourse between monads is fact, then the whole genesis of the concept of a common universe is revealed. If there is an intersubjective intercourse, nothing more is necessary to account for the genesis of the idea of a common world for such an idea must be its immediate consequence. The existence of a common world as the cause of intercourse is unintelligible, just because knowledge of such existence could only arise as an effect of intercourse. The idea of an external world could not arise naturally and spontaneously in a mind cut off from intersubjective intercourse.

Monadology is a rational doctrine from beginning to end just because it works without postulates and is able to reject assumptions. There is no lacuna in its scheme, no stage in its theory which requires a leap in the dark, an appeal to faith, an exhortation to piety, natural or supernatural. It starts with the Cartesian *Cogito*, the immediate identity of consciousness and existence. It shows us that all our ideas, not only the ideas of self and of God and mathematical relations, but all the imagery of sense and all the concepts of the understanding are within us, our own inalienable possession. It shows how in self-consciousness we are given the clue to the interpretation of the nature of reality. To exist is to be active. Pure passivity, absolute inertia, are limits. There is nothing positive in matter conceived as pure inertia, it is non-existence. To do nothing is to be nothing. Yet the limitations to our activity are felt to be real and if they are real they are active and must exist in themselves. There are then other monads and we conceive their in-itself nature on the analogy of our own. The sympathetic or antipathetic responses we receive from these monads enable intersubjective intercourse to be effective to the extent to which the responding activities are attuned to our own. From intersubjective intercourse

arises the concept of the physical world of our daily routine experience.

### 9. PHILOSOPHY AND COMMON-SENSE

Is then the external world of common-sense an illusion and are we called upon to disown it and abandon it? It is an illusion in the same sense in which we now believe the solid earth and moving firmament are illusions. The monadic theory in giving us insight does not change our nature. We may reckon the common-sense world as part of our humanity and we need not do violence to our common-sense view. We are not expected to stand on our heads because optical science can prove that visual images are reversed on the retina. "Les Coperniciens," said Leibniz, "parlent avec els autres hommes du mouvement du soleil." On the other hand, science has everything to gain by a consistent metaphysical theory. The materialism of Epicurus and Lucretius, the Scholasticism of St. Thomas, the scientific realism of Newton and Locke were well enough so long as science meant little more than astronomy and mechanics. Materialism fails completely in biology, and in psychology, and we are coming to see that the reason of its failure is that the sciences of life and mind introduce us to a realm of reality profounder and more elemental than that which we study in physics.

# 10. MONADOLOGY A REALISM WHICH IS THE ANTITHESIS OF MATERIALISM

We are bound to admit however that from the metaphysical standpoint monadology labors with a peculiar difficulty, one which has presented itself at many times and in diverse forms in the history of philosophy. Our only view of the universe is from within. We can never attain the position we are always striving for as a scientific ideal, the vantage point from which we can survey reality with a disinterested and dissociated contemplation. What we distinguish as external to us is not external in any but a purely relative meaning. We have absolutely nothing but our own consciousness of our own activity, self-centered and self-contained, from which and with which to construct our representation of the universe. It seems to us indeed that the reality, the ocean of being which surrounds us, must transcend our finite experience, and yet all we can do is to frame our idea of infinity on this solipsistic foundation. Monadology faces this difficulty, frankly and fearlessly, and instead of recoiling appalled by it, or devising expedients to circumvent it, accepts it. The theory of monads is a realism in the true meaning of the term. The monads are reals in the same meaning in which the atoms of Democritus were reals. The monads are not ideal in the sense that their existence is mind-dependent. They are things—in-themselves. At the same time monadology is the antithesis of materialism. The universe of monadology is a living thing and its constituent elements are living things. There is nothing dead, no substratum of lifeless, mindless stuff. The monads though self-contained enter into com-

pounds. The Cartesians conceived the world as a vast machine which had been set in motion, its large wheels interlinked with and receiving movement from its small wheels, the whole being self-contained. Leibniz conceived the world as living individual every part of which was also an individual, living in its own life and subserving by its activity the organic life of the whole. It is true that in each case it is no more than an analogy but from the standpoint of philosophical theory the monadic conception surpasses the mechanistic conception as St. Augustine's City of God is spiritually nobler than the Apocalyptic vision of a golden-paved jewel-adorned new Jerusalem.

H. WILDON CARR.

UNIVERSITY OF LONDON, ENGLAND.

## FOUNDATIONS OF MATHEMATICS

I SHALL show that Professor Carmichael's fundamental mathematics in the October, 1923, *Monist* (pp. 513-55) is unsound or illogical, or is non-mathematical in any self-consistent sense of the word *mathematical*. Then by citation of the views of other authoritative mathematicians, I shall show his unsound views are also substantially held by those of others, so that therefore orthodox mathematical foundations are unsound.<sup>1</sup> Finally, after completing that frankly destructive criticism, I shall establish sound mathematical foundations.

### I

Carmichael defines postulates or assumptions (p. 513) as certain propositions or "axioms" that are to be left unproved, and when properly chosen constitute the formal bases of mathematical sciences. He then states a conventional mathematical base or foundation as follows (515-16):

Certain logical notions are necessary in the formation of any set of postulates. That of class is one of the most central. Closely connected with this is that of "elements of a class" and the relation of "belonging to a class." For our purpose, we shall take these as primitive notions, into the meaning of which we do not inquire. We may then set up our system of postu-

<sup>1</sup> It does not seem altogether fair to attack Mr. Carmichael for the sins of all mathematicians; but, no doubt, Mr. Carmichael will be able to defend himself, and we prefer to allow our contributors full freedom of discussion.—(Editor.)

lates thus [and Carmichael cites orthodox authorities as a precedent for the same procedure, which citation I omit]:

Let  $S$  be a class the elements of which are denoted by  $A, B, C, \dots$ . We shall need certain undefined sub-classes of  $S$  (classes consisting of a part only of the elements of  $S$ ), any one sub-class of which we shall call an  $m$ -class—thus employing for the undefined sub-class a term which is meaningless in itself. Concerning the elements and  $m$ -classes of  $S$ , we now make the following assumptions: [I omit those specific assumptions or postulates, as we do not need them.]

In this system of postulates (or assumptions), we have two undefined terms, namely, element of  $S$ , and  $m$ -class of  $S$ ; we have also one undefined relation, namely, belonging to a class. So far as the miniature mathematical doctrine based on this system of postulates is concerned, these terms are entirely devoid of specific content except insofar as that content is implied by the postulates themselves. We shall see later that the postulates do, as a matter of fact, very definitely limit the meaning of these terms though they do not make that meaning unique.

As simply a passing comment, I may mention that Carmichael himself, in the last sentence of the passage I have just quoted, destructively criticizes his foundation of mathematics by saying that the meaning of his "terms," of the subjects he talks about, does not become definite or "unique"—which is obviously equivalent to asserting that whatever his mathematics states must be at the very best a double entendre, and in general would be a vague  $n$ -fold uncertainty, ambiguity, or agnosticism. Such uncertainty is strictly *absence* of meaning, rather than any actual meaning or mathematical doctrine.

That sweeping destruction by Carmichael of his own argument is of negligible importance to us here, because it is merely blind destruction that fails to show clearly why his mathematics breaks down. I mention it simply in order to point out that because all his argument is at best a double-entendre, if at any time I attribute to him some statement that has a meaning, by his acknowledged slippery method of speech he can always show that he didn't mean that, but meant something else. In short, strictly speaking, it is not possible to controvert him, or even definitely quote him, because by his own assertion he has really said nothing.

So, strictly, our effort is not to "destroy" his argument, but is to find out why orthodox mathematics logically says nothing, although it uses many words—and often intuitively (or accidentally, from a logical point of view) gets things right. It may, however, be worth mentioning that I have not used a mere slip of his pen to destroy Carmichael's basic mathematics. Repeatedly through his article he modifies his sentences by some such phrase as "in a sense"—and then fails to state what sense, if any, he does mean. And as an even more substantial proof to the same effect, it may be mentioned that Carmichael (514) refers his readers to Keyser's *Mathematical Philosophy* for a fuller statement of his basic meaning, and that in the same October *Monist* (618-34), a third authoritative mathematician, William Benjamin Smith, devotes a number of pages to showing that Keyser in that book is either meaningless or wrong as regards those very fundamentals.

And as a second and last incidental comment, I may discuss Carmichael's verbal ambiguity in the three paragraphs I quoted. In the first paragraph he mentions three "logical notions." In the second paragraph he gives more specific names, or what he calls "terms," to two of those notions—and as an important essential point, which I consider later, introduces and names specifically a *fourth*

notion, sub-class. Then in the first sentence of the third paragraph, he says he has used *two* such names or "terms," and a *third* something (a relation, which apparently is still a "notion"). Thus he has, verbally at least, completely dropped one of the four notions and its name or "term" (class, named S)—which he said in the first paragraph "is one of the most" important. Also, in the second sentence of the third paragraph he seems verbally to have dropped the notion "relation" of the first sentence—unless, indeed, he means now to consider "relation" as being a specific name or "term."<sup>2</sup> The remainder of his article shows that he means (insofar as he means anything) to drop real consideration of relations—which is an important essential error. Therefore, to sum up from this unimportant rhetorical point of view, Carmichael formulates his foundation by introducing four things (one surreptitiously from our point of view, and apparently unnoticed from his point of view), and verbally drops at least one of them, and apparently a second. Presumably he would write this basic part of his argument with at least as much rhetorical carefulness as the other parts. So it obviously is therefore probable that the remainder of his article is as verbally hazy and lacking in rigor as is his base. My observation of the remainder is that it is considerably worse; but that is a negligible point.

We shall now consider the important or essential errors Carmichael makes in that foundation. For convenience, I group them under these three heads: (1) In the orthodox

<sup>2</sup> Carmichael's shift from "notion" to "term" logically amounts to an assertion that in mathematics or logic an idea or notion or thing is equivalent to the name or term or *word* used instead of, or to "point to," that idea or thing. Logically, if the equivalence is *always* thus taken for granted, no error results: there merely ultimately will be and can be no discrimination between actual observation and the expression of that observation—and the commonsense man who indulges in no loose or double entendre talk does not practically need any such discrimination. I briefly show that epistemological point below in Part III. Carmichael apparently uses only vague intuition in his recognition of that basic point—and as we shall see below, frequently falls into error by failing to stick to that commonsense intuition, and by failing to recognize that there are "terms" or words for "relation."

mathematical fashion he gives *no* real or intelligible or actually stated base of mathematics or logic—and as we have seen, naturally winds up in uncertainty or agnosticism, for the simple reason that he starts with that essential ignorance or blank. (2) Even if he had stated a real base, the surreptitious introduction of a fourth “notion” (sub-classes) would have contradicted and destroyed it. (3) His failure to be definite and intelligible as to whether relation either has or is a name or “term” gives the essentially vitiating practical result of capriciously dropping or including relationship. Or, in that respect, orthodox mathematics substantially fails to understand and deal with relationship or “law” or classification, and is therefore fundamentally materialistic—all of which will be shown.

(1) In the first paragraph I quoted above, Carmichael clearly says in effect that it is necessary to start any system of reasoning or mathematics with certain primitive logical notions or ideas, into the meaning of which we do not inquire. (And nowhere else in his article, or in any of orthodox mathematics which I have seen, do I find any essentially different view stated). Apparently, he implies that those primitive notions do have meanings. But we are left quite in the dark as to whether those meanings are self-consistent, and if so how and why, and as to what those meanings might be. It is therefore obvious that those hidden and ignored meanings, if they exist, are the real base of mathematics or logic, and that that base is not stated by Carmichael. He doesn't even assert that it exists: on the contrary, he by logical implication asserts that mathematics is *only* words or expression. So inevitably he ends in uncertainty or agnosticism or intellectual futility. Even schoolboys recognize such inevitableness: *ex nihilo nihil fit*.

(2) But having adopted three notions as constituting an unknown and unproved base, in his second paragraph he introduces a fourth notion, sub-class. I shall show in

Part III that this fourth makes profound changes when introduced into any system based on the first three. But the point here is that Carmichael tacitly takes it for granted that, regardless of the facts of the universe, a mathematician may capriciously add or subtract at any point in his foundations any primitive notion he likes, *without any inquiry* as to whether it is in fact consistent or inconsistent with the other notions. If Carmichael doesn't take that for granted, he at least obviously acted on such a view; and as I have already shown that his words are ambiguous, the only way we have of judging his views is by his actions. And such infinite intellectual or mathematical freedom, or absolute capriciousness regardless of facts, is obviously unlawful, and is not sound mathematics. Carmichael himself, in defining "rigorous thinking" (528), substantially says that it isn't. So if he had started with a real base, he would have destroyed it.

(3) Carmichael's dalliance with "relation" is the same unlawful capriciousness we have just noticed: it simply has become explicit, instead of general and surreptitious. For obviously, the (1) fact or "notion" or relation, and the (2) word *relation* which names that fact (whatever the fact is), are the two things which Carmichael designates respectively as (1) "notion" and as (2) "term." But he refuses to call "relation" a term in his third paragraph above, and thus refuses to call the word *relation* a word, although it plainly is a word (see the footnote above). That absolute verbal capriciousness of saying that a word is a word and simultaneously is not a word gives notice that (1) he proposes, in what he calls mathematics, to ignore such a fact as relationship whenever his capricious fancy dictates, and that (2) he fancies that logic or mathematics is a juggling of words only, in a sort of "spiritual" game or acrobatics, without any particular regard for reality or truth—for the lawful or classifiable facts of the universe. The system of

so-called mathematics given by his article is full of concrete examples of such capriciousness. E. g., he goes so far as to assert (545) that "in a certain sense" the "spirit has thus imposed itself upon nature"—merely admitting that there "appears" to be a real relation between the mathematical "spirit" or capriciousness and the universe. In days of old Jehovah was credited with such capricious power to ignore facts, to alter or suspend law or relation. Then the savage medicine-man and priest claimed similar creative or miraculous power over nature or law. In this modern day we see the mathematician thus claiming that miraculous, materialistic power of Jehovah and the medicine-man. The natural result is the present widespread agnosticism and cynicism.

## II

As was mentioned, Carmichael approves Keyser's *Mathematical Philosophy* (514). On pages 624-6 of the same *Monist*, Professor Smith shows that Keyser in that treatise more or less says that the universe has relations or laws that may not be thus capriciously tampered with, but that Keyser also then contradicts that view, and in actual practice is the up-to-date creative mathematical Jehovah. Smith then goes on, and in plain effect says that mathematics is wholly miraculous—that the mathematician absolutely creates his thinking, even to the extent of creating the moon [which the commonsense man thinks is the moon, and not a frequently newly created mathematical notion]. Of course, Smith keeps contradicting himself verbally, just as he shows Keyser does, and as we have seen Carmichael doing. But I shall not burden the reader with examples of Smith's mathematical self-contradictions; for it is obvious that a

man who talks of the moon which is not the moon will have such troubles in his logic—if, indeed, he may be said to have any logic.

Recently, a mathematical star of the first magnitude has flamed out—*Wittgenstein*, whose *Tractatus Logico-Philosophicus* condenses in eighty pages of text a system of logic or mathematical foundations, a system of philosophy, and of physical science, religion, and ethics. Bertrand Russell writes an Introduction which sums into the extraordinary praise that he can find no obvious logical defect in Wittgenstein's argument, and therefore recommends it to all thinkers. Keyser in a long review of the book (*New York Evening Post*, Aug. 18, 1923) substantially agrees with Russell's estimate; Keyser in a matter of fact way also mentions Wittgenstein (*Sc. Mo.*, Nov., 1923, 492) as being one of the outstanding thinkers or mathematicians of the ages.

Therefore, the interlocking directorate of mathematicians seems to justify us in taking Wittgenstein as the consummate mathematical authority. I myself have high admiration for Wittgenstein. He has the courage of genius in sticking consistently to orthodox mathematical errors—and consequently makes orthodox mathematics sublimely ridiculous. A few technical corrections would make Wittgenstein roughly sound. But as is perhaps already obvious, it would scarcely be possible to unscramble such a hash of contradictions as Carmichael typically makes.

Wittgenstein is explicit that orthodox mathematics has no base or real knowledge, whereas Carmichael is vague and continually self-contradictory about such fundamental agnosticism. Wittgenstein concludes that only physical or material "facts," or what Carmichael calls "terms," may be expressed or dealt with positively by mathematics or logic—that we can't even intelligibly talk about such things as principles or foundations or what Carmichael calls

"notions," or about meanings or real sense, cause and effect, law, ethics or morality, God, and truth. Russell, whom Smith calls brave and "more militant than his peers" (627), in his Introduction timidly hesitates to accept that dropping of sense or meaning from mathematics, on the ground that although he can find no logical error, still Wittgenstein "manages to say a good deal about what can not be said." By thus refusing to go where his logic leads, Russell clearly expresses contempt for his own logic or mathematics. And for an authoritative mathematician thus timidly to throw logic to the winds is obviously the moral equivalent of a mother's throwing her babe to the wolves to save herself—a procedure of a character that is not inexpressible, but is conventionally unprintable.

Wittgenstein does not capriciously introduce new "notions" openly, as does Carmichael. But he makes the fundamental orthodox error that is basically equivalent to such capriciousness, by dealing with relationship sometimes as a (1) "term," sometimes as a (2) "notion," and in substantial practice by (3) ignoring or dropping it entirely—all three ways being wrong, as we shall see in Part III. As we have noticed, Carmichael deals with relationship at his "creative" fancy. Wittgenstein is not so crudely capricious as that. In fact, Wittgenstein at least verbally repudiates all such blatant claims of creative, supernatural divinity by stating (6.031) that "classes are altogether superfluous in mathematics"—which denies not only mathematicians' miraculous ability to tamper with law or relationship, but denies the very existence of *any* relationship or "belonging to a class."<sup>3</sup>

<sup>3</sup> Professional mathematicians may desire the following brief citation of Wittgenstein's varying views to the effect just stated: (1) He ignores the fact that mathematical signs, such as the equality sign, are really relationship words; in 4.242 and 5.534 he substantially denies that the equality sign has any meaning—although obviously its general meaning is "is." Thus according to him mathematics is a language without ultimate relationship words—so naturally is without "sense" or meaning, but is merely a loose chaotic list of nouns or *x*'s without any verbs or "is's." That amounts to the 6.031 cited above:

Thus it appears that authoritative mathematicians are in practical agreement (1) that there is no real base to mathematics or logic, and (2) that mathematicians are endowed with supernatural creative power, and hence need not in practice bother to notice any such thing as truth or relationship or law or reality. Wittgenstein courageously goes to the end of that irrationalistic, agnostic, skeptic materialism, and consistently with such nonsense asserts that a belief in causality [i. e., in law, relation, value or worth-whileness of any sort] *constitutes superstition*. With even more sublime mathematical genius, he agrees that everything he has said is really meaningless and must logically be discarded, and he therefore subsides into logically ultimate silence with the final conclusion that we must not speak about what we do not know. As a simple truism, we certainly ought not to.

Therefore, their own highest authority has correctly reduced orthodox mathematicians logically to eternal silence. If one speaks hereafter by that very fact he logically repudiates orthodox mathematics.

### III

As the mathematicians themselves have thus wiped out orthodox mathematics, let us fill the void by formulating true mathematics. "Half" of sound science obviously is "classes are altogether superfluous." (2) Wittgenstein more or less asserts in 4.241, and at least usually acts on the view, that one fact or "term," although it is related to another, may logically be absolutely dropped. He thus endows the logician or mathematician with power *absolutely to annihilate terms or facts*, and then necessarily or truistically, to create them (cf. his aphorism, 5.551). We have seen Carmichael do such creating or ignoring of law or relation, by introducing sub-classes. Keyser flatly asserts that power by asserting the primitive postulate that "if  $p$  implies  $q$ , and  $p$  is true, then  $p$  may be dropped and  $q$  asserted," and calls it "an exceedingly subtle principle introduced . . . by Peano." (*Science Hist. of the Universe*, VIII, 203). But verbally contradicting that materialistic pseudo principle, Wittgenstein in 5.452 seems to hold Russell responsible for such witch-doctor propositions, and to repudiate them himself. Therefore, Wittgenstein probably actually agrees with the sound mathematics I outline in Part III, but simply does not know how to formulate it.

*observation*, or "notions" or facts; the other indispensable, inseparable "half" is *expression* or communication, or language, logic, mathematics, "terms," or the systematizing or classifying of those facts. Even an earthworm, on being touched, expresses knowledge by a gesture, a primitive muscular "word" or act. We need a sound language or logic, we need mathematics in the true sense, if we are to have any knowledge or science beyond the experience and expression of animals. The lack of fundamental soundness in current mathematics has filled the world with agnosticism and materialism—with uncertainty, futility, timidity, and wordy quarrels, with cynicism, and grabbing of things. We saw even the "militant" Russell lapse into the bitter cynicism and timidity of throwing his beloved mathematics to the wolves.

We are dealing here primarily with mathematics or words—with any sort of symbols or "expressions" that point to or mean observations. It is obvious, as being a simple truism, that if we never use symbols except to point to something actually observed, then, when we have learned or observed such interconnection between certain words and certain corresponding facts (have "learned a vocabulary"), we may formally or in principle deal with words as with facts or observations. That truism is the substance of epistemology—shows what "knowledge" or meaning is, with respect to its expression. The truism is the base under mathematics—under all language or logic. It "assumes," or postulates or leaves unproved, absolutely nothing—unless the fact that we observe the universe is called an "assumption," in which case as a direct truism *everything* is an assumption, and "assumption" then means identically the same as our usual word "knowledge," and we should have to invent another word to mean what we now mean by *assumption*. In footnote 1 we saw that the truism was vaguely intuitively asserted by Carmichael, who then failed,

as we further saw all orthodox mathematicians failing, to follow it steadily.

Mathematics thus either must be based directly upon the real universe, or mathematics does not and can not exist. For if a given alleged word does not directly or indirectly (i. e., "negatively" point to some thing that is observed or "has meaning," then as a truism that word shows or indicates absolutely nothing, can not acquire meaning (can not refer to any experience), and simply does not exist as a word—is not a word, and is no part of a sound mathematics or language.

Therefore, if we observe how words or symbols are used in those cases where they *are* actually used—where they give us an idea or feeling that they "work," or mean something, instead of winding up in ignorance or agnosticism—then we shall, as a further absolute truism, find both the basic principle of logic, and simultaneously the basic truth about the real universe which those useful words "express" or point out. We are well aware that ordinarily our talk serves us with at least some satisfaction. Even orthodox mathematics itself usually is satisfied and sure that its discussions are not only consistent, but really mean something and are in some way useful. And most orthodox mathematics actually is sound: most of the practical views Carmichael gives in his article are more or less correct. It is when mathematicians deal with the foundations of mathematics that they get into the hopeless muddle we have noticed, and solemnly argue the nonsense that the moon is not the moon, or that superstition is the belief in cause and effect or relationship.

By observing language which works, we can see that there are three kinds of words—or *phrases* (e. g., adjectives and adverbs are rationally incomplete words that are not any of those three kinds, but in order to become genu-

ine words join on to other words to make phrases of those kinds). The three kinds of words are as follows:

(1) There are words like *all*, *everything*, *class*, *infinity* (or *zero*, which is the indirect or "negative" form of *infinity*—indirectly asserts infinity by pointing "away from" it), *energy*, *universe*, *God (the Father)*, which words indicate the total universe or reality. We may call those One words—as they name an ultimate "One" or unification. They are more or less what Carmichael calls "class" and also "notions"—but then like an ostrich sticking its head into the sand, he declines to admit that they are words, or that he uses words for that unity.

(2) There are words like *this*, *that*, *electron*, *God (the Son)*, *twenty-second thing*, *part*, which indicate parts of the universe or One. We may call those Many words—as they mean some discriminated part or thing in the universe—considered-as-Many-parts. Carmichael more or less calls these words elements of a class, or "terms." They are in effect the only sort of words that Wittgenstein (or any *strict* orthodox mathematician) acknowledges—so that actually he talks a language like the hen, who by clucking and pecking at a bit of food says, "This, this, this," to her chicks, and *says* nothing else. Naturally Wittgenstein must conclude that he has not really said anything in the ordinary sense of *said*. Mathematics obviously never will say anything so long as it confines its language to the elementary, primitive clucks of a hen or the gestures of an earth-worm.

(3) And third and last, there are words like *force*, *love* (with its indirect form, *hatred*), *fatherhood*, *God (the Holy Ghost)*, *being*, *Being*, *is*, *of*, *life*, *personal*, *redness*, *truth*, *spirit*, *beauty*, *value*, *system*, *order*, *science*, *number*, *function*,  $+$ ,  $-$ ,  $\times$ ,  $\div$ ,  $=$ , *equals*, *time*, *space*. We may call those relationship words—as they indicate or assert that

the parts of the Many are related together, or joined together into a One or universe.

Obviously, One words flatly contradict Many words: the universe or reality plainly can not be a One or unit, and also be a Many, or absolutely split into pieces. It is that glaring contradiction which causes Carmichael and mathematicians in general to be so vague and evasive about their One "notions" or "classes," and about the contradictory Many "terms." The contradiction exists, and mathematicians don't know the solution, don't face the difficulty, and naturally wind up in futile evasion, agnosticism, and self-destructive ignorance.

The first thing that puzzles those mathematicians, the only real difficulty, about the absolute solution of that contradiction, is that the solution is so excessively simple—as follows: relationship words assert that the Many words are really joined, and obviously a really-joined-together-Many is a One. Or, to say that simple solution over again, sound language says that the related-Many *is* the One—which is obviously true, and non-contradictory. Or, Many words do contradict One words, but relationship words then re-contradict the Many-ness, making it really One. Or, in soundly consistent language, there are *two* verbal contradictions, which obviously cancel each other.

And that is the total mechanism of sound mathematics or language. In every sentence which means something, we say that Many parts are related together and *are* the One; or the related-Many *is* the One; or  $f(x) = 0$  (or  $\infty$ ); or specifically, in everyday terms, *This + That + ... = The One*, or *This... × That... = Meaning* (where the symbol ... means a continuously related sequence or regress—and obviously indicates that all our intelligible sentences *imply* the total infinite universe or reality or God).

The second thing that puzzles orthodox mathematicians about that truly logical sentence is that such a sentence is

an absolute tautology; i. e., to say that *The related-Many* = *The One* is to say *The One* = *The One*. Obviously, we *must* say everything twice *if* we are to use that double self-canceling contradiction, and thus have a language that *positively* expresses our knowledge or observation, instead of merely having Wittgenstein's hen clucks. Therefore, *logical* proof, or *expressed* proof (as contrasted with real proof), obviously is such tautology repetition, or truistic re-saying, of everything we say. By such tautology or *truism* we simply say (and only by such truism can we say) that we have observed the real meaning both as a One and also as Many parts that work together, are observed together, are related together or do "mean" something. *Real* proof is the observing itself, the living of life in and as an inseparable part of that universe or reality—not the saying.

The third and last thing of much importance that puzzles and confuses orthodox mathematicians is as to the nature of what Carmichael surreptitiously introduces as sub-classes. The difficulty with that puzzle is that its solution is as simple as the other solution—so simple, in fact, that it is sub-consciously known and correctly used by every commonsense man, and usually by the mathematicians themselves. Obviously, ordinarily we do not discuss the *total* universe: usually we simply talk about our more or less immediate environment, which is only a part of the whole universe. In such ordinary talk we do however clearly treat that *part* of the infinite universe *as if* it were a One or a whole. For the purpose of the talk, that environment or "practical" universe has the *form* or mechanics of a One, is a single unit that *is* the sum of the related-Many we distinguish in it, and hence is *logically* a One, which we may name a "standard One" or "standard universe" or a "sub-class." We can then, as a simple truism, go on and use that standard One with absolute *logical* consist-

ency, in the same way as we use the whole One. *But*, in that case, although we shall *qualitatively*, or in principle, or *logically*, or formally be sound, we shall not be *quantitatively* accurate. For it is an obvious fact that all the remainder of the universe is acting on that standard universe all the time we are talking about it, changing it continually, so that *in fact*, in *quantity*, it does not remain the same One, does not continue to have the same unit sum, even though we formally consider it steady, or an eternal unit, or "static." Therefore, all use of standard universes or sub-classes *quantitatively ignores*, or *drops out of the immediate discussion* (does not drop absolutely—we *can't* abolish law or relationship, so we make no capricious effort to do so), the rest of the universe or One or class, and hence quantitatively neglects *some facts which really do exist or continue to be related*; and therefore the use of standard universes is *inaccurate* talk, and can be relied upon *only* when the remainder of the universe is either tacitly or explicitly considered.

Or, changing the point of view, our sound tautological or truistic logic is obviously reasoning or talking "in a circle"—all truism is a verbal "circle." That circular logic is sound. As a glaring truism, it is the only sort of reasoning that is possible, when the total universe is included—as we can't get outside the universe: we started with that and with nothing more. That is simply the epistemological truism we started with; and we have now circled back to it, proving it. But when only a part of the universe is formally included, such circular logic becomes a narrow, or vicious, or finite circle of reasoning, and is *quantitatively* inaccurate in that it neglects some facts, but is formally or *qualitatively* right (provided, of course, the double and self-canceling contradiction of sound logic is *always* included).

The practical importance of "standard universes" lies in the fact that our ordinary "science" deals only with parts

of the whole universe. E. g., even in the science of logic or mathematics we do not *actually* write down *all* the infinite possible symbols, and all the unending sorts of sound languages. Hence, although we may easily and simply be formally or in principle absolutely sound in mathematics and other sciences, necessarily we are *quantitatively inexact*. I. e., *there can be no exact science*. Obviously, it would require an infinity of symbols to state exactly the value (the *comparative* "measure" or relation) of *any* Many part—and even then it would change in the next instant. Any man who asserts an exact science, or asserts such details as the precise weight of electrons "all alike," or exact atoms of which there are a definite number of kinds, or exact or absolute quanta, etc., simply advertises his fundamental ignorance and materialism. For a Many or material part is not thus absolute or exact—the existence of relationship denies any such absoluteness or importance or "value" to a part. But any man with a modicum of commonsense, not only can, but actually ordinarily intuitively does, grasp the absolutely true quality or principle of the universe or reality—and positively expresses it whenever he says anything intelligibly.\*

\*I may briefly point out three technical points that are important to mathematicians—only implying the proof (for explicit proof, see my book, *Universe*). (1) *Zero* and *infinity* are One words, and are not ordinary numbers or "elements" or Many terms. An ordinary number names a part; infinity (and zero, indirectly) names the whole, the sum of the parts. For mathematicians to say that infinity and zero are numbers, as they do, is to say that the whole or One is the part, or a Many. If zero and infinity be accepted for what they are, the much confused and erroneous orthodox theory of number clear up easily and simply. (2) The mathematical signs  $\times$ ,  $+$ ,  $=$ , etc., are relationship words, and should *always* be recognized as indispensable logical parts, that can not be capriciously ignored or dropped, or *duplicated* (as when the *square root* of a *minus* quantity is orthodoxly supposed to have a meaning of itself, etc.: a relationship of a relationship is meaningless and self-contradictory; cf., motherhood of brotherhood). As a matter of fact, there exists *only one* relationship, that of continuity or *ultimate* identity. All other relationship words are *implicit* assertions of that one relationship (of the ultimate truism, *The related-Many = The One*), and must in definitely rigorous logic (i. e., in mathematics) be made *explicit* by the final use of the equality or identity sign—i. e., by finally reducing the statement *explicitly* to a tautology or identity or truism or "equation." The failure of orthodox mathematics to accept relation as a word results in orthodox mathematics being a *dualism*—which is an incomplete or non-expressive language containing or recognizing only (1) One words and (2) Many

And the logical proof of the soundness of that commonsense logic is the truistic summing up of the foregoing, thus: that a failure to say, in any statement, that the related-Many is the One constitutes unintelligibility, and is the only logical error. Or more specifically, a failure to use all three forms of words, in their proper double and canceling contradiction, is logical or verbal error. Or still more specifically, to drop one sort of word, or (what amounts to the same thing) to confuse one sort of word with another sort, is the only logical error.

Of course, the *real* proof of the foregoing commonsense logic consists, for each man, of his own observation of its soundness. If he hasn't got the commonsense of the average man and therefore can not see that soundness, there is no positive or direct way of *making* him see that soundness. We can keep pointing out the facts to him in different ways—keep trying to educate him. But possibly his brain will be too defective to see it. So until we learn some direct method of repairing defective brains, there is no positive or direct way of teaching sound mathematics.

words, that naturally contradict and really destroy each other when thus dualistically used without relationship words. That incomplete dualism or half-baked language therefore necessarily gives the futility or agnosticism we have noticed. And Wittgenstein, noting the absolute unworkableness of that usual mathematical dualism, tries to correct it by dropping One words—and winds up even worse, with only Many words, or hen talk. Mystics, analogously to Wittgenstein, try to talk a non-working language of only One words—try to talk all in infinities. Many mathematicians become so disgusted with the unworkable hen-talk, that for a rest and change they revert to mysticism, which similarly has only one kind of words. (3) The trinity language with the *double* contradiction, needing *three* sorts of words, which I have described above, is our everyday language or logic, and is the *most economical positive language*. I. e., we must have *at least* three sorts of words in order *positively* to say something. Obviously, we may make any number of *sound* languages, the criterion being that each must contain an *even* and hence canceling number of contradictions. So we could have less economical, but sound, languages with 5, 7, 9, etc., different kinds of words. Orthodox mathematicians (without recognizing what they were doing) have already formulated some such languages, calling them transfinite numbers, and non-Euclidean spaces. Also, Einstein's relativity is vaguely and confusedly such a language. But as a simple truism, those languages, if translated into our everyday trinity language, mean, and also verbally sav. just what we ordinarily sav. E. g., "four dimensioned space" is a "foreign" phrase, which when rationally translated into our everyday language, becomes the phrase, and means, "three dimensioned space."

And the validity of that truistic or identic logic or mathematics depends fundamentally upon the soundness of the epistemology which I logically proved at the beginning of this Part III. The commonsense man at once observes that when he sees or knows or "means" something he can understandably express it to a man with a normal brain. Epistemology is the science which asserts and shows that truism—shows that the whole of facts is ultimately related identically with the whole of sound expression. We saw in footnote 1 that mathematicians intuitively assumed that foundation instead of stating it and proving it. Conventional science also obviously assumes it when it says that science is knowledge or observed facts, "duly arranged" or ordered—i. e., *consistently* expressed. Sound knowledge (the first step of which is technically named epistemology) assumes nothing, but proves that "half" of science is observation or experience, and the other inseparable "half" is expression or mathematics. Therefore, sound mathematics in the widest sense obviously should begin at the beginning, and by at least outlining epistemology show that all observations or facts agree with, or are really related to, the sound expression or mathematics-itself, in the more narrow technical meaning. Orthodox mathematics omits both the real proof and the logical proof—and thus both practically and logically has no base, no foundation, and *strictly* no sense. There is no space here to give *real* proof of that ultimate base: I have stated it at some length in *Universe*, which definitely gives that "concrete" or actual unification of science, by pointing out all the typical observed facts. This article deals primarily with words.

Basic science or epistemology, then, consists of (1) the seeing or consciousness of the universe itself, and of (2) mathematics or the arbitrary expression of it. When we come to examine (1) the actual consciousness or observation, we see that it is continuous—or that the One, or *mon-*

*ism in the sense of unbroken relationship* or action-reaction or cause-effect, is true or is reality. When we come to examine (2) expression or mathematics, we see that we discriminate parts of the Many unendingly or relatedly—or we see that that related-Many or related-pluralism, or *infinite* pluralism is the truth or reality.

In short, we have the same truism or tautology as before: infinite pluralism (or reality from the point of view of mathematics) *is* monism (or reality from the point of view of the actual observed *universe* or One). Thus a truly sound epistemology or basic science reconciles—definitely identifies—mathematics with experimental or inductive “science”; and shows that the so-called riddle of the universe is merely the basic puzzle of language or logic. The same truism, or verbal “paradox” of double contradiction, easily absolutely solves all other logical or qualitative problems. And as we saw, *no* quantitative problem is exactly or absolutely soluble—except by actually “living” it.

To make that general solution of everything more intelligible, I shall now briefly show the two concrete evils that result from the fact that orthodox mathematical science tends to ignore both (1) the One, and (2) relationship—especially relationship.

(1) *Religion* is a word that ordinarily is used rather indiscriminately in each of the three trinity forms—just as the word *God* often means more or less the whole Trinity. (Incidentally, our vocabulary contains many words that may be used in each one of the three forms; the usage of a word in a given place of course determines its meaning and logical character—which is never exact or absolute: there is no *perfect* logic, in a concrete or factual sense.) We know so well, intuitively, what religion is, that I shall not use the considerable space needed to discriminate clearly those three meanings (the One, or mystic meaning; the Many, or theology and the science of ethics and Biblical

interpretation; relationship, or love and faith). I shall simply say roughly that religion consists of seeing the universe as an organized or related whole (as a "personal" God, if you prefer that name, which obviously means the essential One); and therefore say that religion naturally or truistically consists of having absolute knowledge that the One will work lawfully or "justly" or with unbreakable relationship or "force" or "love." Such observation of the One (with simultaneous observation or "acceptance" of whatever standard One, or God the Son, you prefer to select, if any, as a sort of practically graspable, but finite, high standard for yourself)—such observation of the One gives various degrees of nervous contentment, or self-confidence and courage, or activity, or ecstasy—depending on circumstances. And such observation or certain knowledge is *faith* (in the real sense—not in the self-contradictory Modernist or "scientific" sense of a "hypothesis," which is a guess at a law or relation: a truism or law is either known or not known, and it is merely silly to talk of guessing at it: we guess at *quantities*). But as we have seen, orthodox mathematics omits, or even denies, the observation "half," or One universe, and therefore is necessarily agnostic or meaningless, or has no faith, no observation, no real courage, no religion—and no possibility of rationally acquiring a religion so long as it persists in that error. As orthodox mathematics is accepted as basic by so-called exact science, modern exact science is therefore irreligious, with all the truistic immorality which that necessarily implies. The leading inexact scientists (biologists, psychologists, etc.—the naturalists and humanists), such as Jordan, Ritter, Patten, Dewey, Hall, Laughlin, Chamberlin, have not been perverted by modern mathematics; but even some naturalists, such as Bateson and Pearson, by trying to ape mathematicians have been perverted. I know of only one writer on the philosophy of mathematics who has

been able enough and strong enough to resist accepting the errors of modern mathematics: McCormack.

(2) And as orthodox mathematics substantially follows Wittgenstein in dropping relationship, and accepting definitely only the material Many, modern mathematical science is materialistic or inhuman. Such science asserts in practice that there is no such thing as relationship, or law, or spirit. It is true that many mathematicians and exact scientists do much talking about "spirit." But that is a mere smoke-screen of words, which seems chiefly to hide from *themselves* the uncomfortable fact that their professional doctrines are nonsensical or meaningless. In fact, by going preposterously far with that pious make-believe talk, and asserting that they are absolute spiritual creators, or medicine-men controllers of relationship, they practically demonstrate that they do not understand their talk about relationship, or the organization that is "personality" or is human. Such materialism has become so definite among mathematical scientists that the word *evolution*, which in sound biology simply means continuity or universal relationship, has been pervertedly interpreted by such mathematical scientists as meaning a continual material addition, or growth up, up, up, like a profiteer's piling up money. Incidentally that materialistic interpretation of evolution flatly contradicts the second law of thermodynamics, which also ignores relations, and says the universe is material going down, down, down. So the exact scientist may be a prophet of absolute doom one minute, and an instigator of profiteering the next—and be wrong both times.

Thus mathematicians have so infected science with error, that exact science now deserves the condemnation which is indiscriminately given to all of science by the Fundamentalists. I think the most serious troubles of men now are the results of irreligion, in the form of that agnosticism and materialism. And as I have proved above, that

irreligion is taught directly and basicaly by mathematical scientists. The mathematicians have been the court of last resort; people have to a large extent relied upon them to lead them aright in fundamental knowledge—to show them how to see and then to do. And mathematicians have basicaly failed, and led them wrong. The Fundamentalists wish to correct the mistake by substantially reverting entirely to seeing the One (to mysticism), and in some cases by trying to abolish efforts to discriminate the Many. They are really trying to revert to infancy, where we see the universe as a continuous One or mystic blur, and where we have no positive speech or much discrimination, and hence control, of the Many—not enough, in fact, to keep us alive of our own efforts. Specifically, the Fundamentalists would set up the Bible as stating the inerrant or exact truth. We have already seen that there can be no exact or inerrant science, or statement of the truth. The Fundamentalists have fallen into precisely the same basic error as the mathematicians when they fancy that any finite set of words, such as the Bible, can be inerrant. The reported doctrines of Christ do give the qualitative truth excellently. But Christ himself is reported to have acknowledged that he did not state the qualitative solution of basic epistemology, by stating that certain sayings were for those with understanding, or for those who could receive them.

As both the Fundamentalists and the mathematical scientists are wrong about the basic essentials, intelligent men must, if they are to survive, turn away from both those mistaken set of specialists and leaders, and study, and practice as well as may be, the simple truth.

S. KLYCE.

WINCHESTER, MASSACHUSETTS.

## THE ROLE OF SEQUENCES IN OUR SEARCH FOR TRUTH

NEWTON'S laws of motion seemed to us to be the final word in the explanation of physical phenomena. But now recently have appeared Einstein's laws of relativity, which (if verified) explain all the phenomena taken care of by Newton's laws and some not accounted for by the latter. Einstein's formulae reduce to those of Newton for motions with velocities that are infinitesimally small when compared with the velocity of light; but at first glance the two sets of formulae look very different. If we draw a polygon  $A_1$  and say that the interior of  $A_1$  shall represent all the phenomena explained by Newton's laws, and draw a similar polygon  $A_2$  for Einstein's laws, we see that  $A_1$  will be entirely within  $A_2$ .

We have no reason to suppose that we may not later find phenomena unexplained by relativity, and then some scientist may discover further laws that will include both Newton's and Einstein's as special cases. If we draw a polygon  $A_3$  for the phenomena explained by these hypothetical laws, we shall have  $A_1$  inside  $A_2$  and  $A_2$  inside  $A_3$ . Then perhaps we may stop at  $A_3$ , because the corresponding laws explain all the phenomena (explainable by laws of motion) that ever can come within the ken of human beings. Perhaps we may have to consider polygons  $A_4$ ,  $A_5$ , etc., to a finite number of such polygons; or finally we may have a never-ending succession of such polygons. We call such a succession of polygons (whether ending or never-ending) a sequence, and we call its separate polygons the terms of this sequence. In this paper we shall discuss these sequences of polygons and their corresponding sequences whose terms are sets of laws.

We find everywhere in our search for truth such sequences of laws and of polygons as the above. The Ptolemaic geocentric system of astronomy (for example) explained many phenomena fully as well as does the Copernican system, but the latter explains these more easily and

also handles many more phenomena than the Ptolemaic can handle. So also we have the electron theory supplanting or supplementing previous physical theories about the constitution of matter; the various theories of the propagation of sound and of light, etc., etc. In the realm of social sciences we find similar sequences. We find them even in theology. For example the theologian speaks of the love of God for mankind as an explanation of various phenomena. This idea, if analyzed, discloses a sequence of concepts of greater and greater loves (each including the preceding) starting with the love of man for man and ending so he believes in the love of God for man.

Every such sequence, no matter where it occurs, belongs to one of the following classes:

(A) It may have a finite number of terms (like the sequence of the positive integers from 1 to 10 inclusive), in which case it has a last term and is called a finite sequence.

(B) It may not have a finite number of terms, in which case it is called an infinite sequence, and it belongs to one of the following sub-classes:

1. Its various terms may not approach indefinitely near some limiting term, in which case it is called divergent. (Thus we have in algebra the divergent sequence  $x=1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}$ —and other divergent sequences.)

2. Its various terms may approach a limit that is easy to ascertain from the terms themselves. (Thus  $x=1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{4}$ —is a sequence whose limit is 2.)

3. It may have a limit that is not at all like the terms of the sequence. (Thus in the following paragraphs we find that the circle is very unlike any term of the sequence of inscribed regular polygons of which it is the limit.)

4. It may have a limit that can never be described or written down in its entirety. (Thus the sequence  $x=3, 3.1, 3.14$ —has the limit  $\pi$ , the ratio of the circumference of a circle to its diameter. The value of  $\pi$  has been computed to seven hundred decimal places, but it can never be expressed in its entirety. Also a more technical example is that of the sequence of polynomials in powers of  $x$  whose limit is  $\sin x$ . This limit, called an infinite series, can never be written out fully and completely.)

Let us consider for a moment two more examples of sequences. If we inscribe in a circle a regular polygon of five sides (say), the area of this polygon is less than the area of the circle. If we keep increasing the number of sides of our regular inscribed polygon, each side becomes smaller and the area of each successive polygon is less than, but more and more nearly equal to, the area of the circle. The inscribed polygons approach the circle as a limit as we increase indefinitely the number of their sides. But the circle is not a polygon. Many errors crept into geometry until geometers realized this fact. Again there is a formula much used in bridge engineering. Recently bridges of longer and longer spans (like the Quebec bridge) have been con-

structed. The formula has needed revision because it does not hold true for such long spans. It has been found that the formula is one of the polynomials of a sequence whose limit is an infinite series. So now for spans of greater length, the engineers must take polynomials further along in this sequence.

Let us consider now the effect of these sequences on the belief or disbelief in the existence of absolute truth. We have no means of proving logically to what class any sequence of laws we are studying belongs, because we cannot look into the future and see if any new phenomena will arise that are inexplicable by the existing laws of this sequence, and that is the only criterion by which to judge this sequence. The theologian, when he discusses the love of God for man, may be embarking on an infinite sequence of loves that has no limiting term. And even if there is a limit to his sequence, how can he ascertain the form of this limit? An absolutist philosopher will assert that the polygons  $A_1$ ,  $A_2$ ,  $A_3$  approach the circle of phenomena to be explained by the corresponding laws; and that the sequence of laws has a limit, which he calls the absolutely true law. He will say that each successive law is a closer and closer approximation to this limit. He still must strive to ascertain what sort of a limit his sequence possesses. He is constantly being disappointed when new phenomena prove his supposedly absolutely true laws (like Newton's laws of motion) to be only members of sequences of laws. He has at his hand no other weapons of research than those used by the believer in the relativity of truth; and the way seems just as long, except that he believes he sees his goal of absolute truth shining in the distance. The relativist philosopher, on the other hand, sees the seemingly endless path of research ahead of him; and he does not believe these sequences (he is studying) have limits. He does not call each improved law a closer approximation to any hypothetical absolutely true law; but he takes it for what it appears to him, namely as another term further along in an apparently infinite sequence, where each succeeding term is a better explanation of the phenomena than all the preceding terms.

ALAN D. CAMPBELL.

does  
that  
whose  
eater  
along

n the  
We  
any  
can-  
a will  
this  
udge  
love  
uence  
e is a  
f this  
poly-  
to be  
uence  
law.  
closer  
scer-  
con-  
re his  
ys of  
e has  
used  
eems  
abso-  
iloso-  
th of  
these  
t call  
theti-  
pears  
ppar-  
is a  
eding

LL.